Fluid Flow around an Elliptic Cylinder using Finite Element Method

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Abstract

We present the Finite Element Method (FEM) to compute the solutions of Laplace/ Poisson equations in terms of stream function. First fluid flow equations for an inviscid incompressible fluid are derived in terms of velocity potential and stream functions. A boundary value problem (BVP) governed by Laplace/Poisson's equations with Dirichlet and Neumann boundary conditions is considered. We use the triangular elements to obtain the FEM solution. As a specific example, we consider a BVP that arises from an inviscid incompressible flow around an elliptic cylinder.

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MATHEMATICAL FORMULATION

Two dimensional boundary value problems given by Laplace's and Poisson's equations

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ and } \nabla^2 u + f = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f = 0 \quad (1)$$

occur very frequently in applied mathematics, science, and engineering ([1], [2], [3], [4]). Few physical situations which can have models involving these equations include: incompressible inviscid fluid flow, steady state heat conduction problems, diffusion flow in porous media, torsion problems in solid mechanics, electrostatic potentials, gravitational or Newtonian potentials and magnetostatics.

FLUID FLOW EQUATIONS

Equation of continuity is given by ([4])

$$\frac{\partial \rho}{\partial t} + \text{div}\left(\rho \vec{q}\right) = 0 \quad (2)$$

Here $\vec{q}=(u, v)$ is the velocity, ρ is the density of the fluid. If the fluid is incompressible, then ρ is constant, and the equation of continuity becomes div $(\vec{q}) = 0$. If the motion is irrotational, we have curl $\vec{q} = \vec{0}$. For this case, there exist a scalar function ϕ ; called velocity potential function, such that $\vec{q}$ = grad ϕ. The velocity components $(u; v)$ of $\vec{q}$ are given by $u = \frac{\partial \phi}{\partial x}$ and $v = \frac{\partial \phi}{\partial y}$.

Thus ϕ satisfies the Laplace equation

$$\operatorname{div}(\operatorname{grad}\phi) = \nabla.(\nabla\phi) = \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \qquad (3)$$

Stream function, ψ and velocity potential ϕ are related by $u = \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}$

and $v = \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}$, so that ψ satisfies the Laplace equation

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \qquad (4)$$

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PROBLEM STATEMENT

We consider the following boundary value problem (BVP)

$$\nabla^2 u(x, y) + f(x, y) = 0 \qquad \text{in } D \qquad (5)$$

$$u = g(s) \qquad \text{on } \Gamma_1 \qquad (6)$$

$$\frac{\partial u}{\partial n} + \alpha(s)u = h(s) \qquad \text{on } \Gamma_2 \qquad (7)$$

where D is the interior of the domain, and Γ_1 and Γ_2 constitute the boundary Γ. The boundary condition on Γ_1 is referred to as Dirichlet (Essential) boundary condition. The boundary condition on Γ_2 is known as Robins boundary condition. If $\alpha = 0$, equation (7) becomes Neumann (Natural) boundary condition.

SOLUTION BY FINITE ELEMENT METHOD

We employ the following steps to solve this BVP using finite element method (FEM):

1. Domain discretization.
2. Weak formulation.
3. Derive interpolation function.
4. Computation of element matrices and boundary integrals.
5. Assembly of element equations.
6. Imposition of biundary conditions.
7. Solve the matrix equation.
8. Postprocessing of the solution.

DOMAIN DISCRETIZATION

First we discretize the whole domain by dividing it into a finite number of elements. That is why this method is called Finite Element Method (FEM). Here we use linear triangular elements. These elements are constructed by joining the nodes. We use three nodes for each triangular element. Another common element used in FEM is quadrilateral element.

We need to number the elements and nodes for implementation. Now we consider a particular element e with its area D^e and boundary Γ^e.

WEAK FORMULATION

The weak form is derived from a weighted integral statement. First, we multiply the governing equation (5) by a test function *w(x,y);* which has first order partial derivatives, and then integrate the resulting equation over the element e to distribute the differentiation evenly between *u* and *w* as follows:

$$\iint_{D^e} w(x,y)\left[\nabla^2 u(x,y) + f(x,y)\right] dA = 0 \tag{8}$$

To obtain the weak formulation, we use the following theorem

$$\iint_{D^e} \operatorname{div} \vec{q}\, dA = \int_{\Gamma^e} \vec{n}.\vec{q}\, ds \tag{9}$$

where $\vec{q}$ is a vector field and the line integral is evaluated in a counter-clockwise sense around the bounding curve Γ^e with the normal $\vec{n} = < n_x, n_y >$ pointing outward. Equations (8) and (9) yield

$$\iint_{D^e} \left(\frac{\partial w}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial u}{\partial y} \right) dA = \int_{\Gamma^e} w\, d_n ds + \iint_{D^e} w\, f\, dA \tag{10}$$

Here $d_n = n_x \dfrac{\partial u}{\partial x} + n_y \dfrac{\partial u}{\partial y}$. The first integral on the right is a line integral taken in the counter-clockwise direction around the boundary Γ^e.

INTERPOLATION FUNCTIONS AND COMPUTATIONS OF A_{ki} AND b_k

Now *u(x, y)* is approximated over a typical element e by

$$u(x,y) \approx U^e(x,y) = \sum_{j=1}^{n} u_j^e N_j^e(x,y) \tag{11}$$

where u_j^e is the value of U^e at the *j*th node (x_j, y_j) of the element e, and N_j^e *(x,y)* is the interpolation function with the properties

$$N_i^e(x_j, y_j) = \delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \tag{12}$$

and

$$\sum_{i=1}^{n} N_i^e(x,y) = 1. \quad \sum_{i=1}^{n} \frac{\partial N_i^e(x,y)}{\partial x} = 0, \quad \sum_{i=1}^{n} \frac{\partial N_i^e(x,y)}{\partial y} = 0.$$

Substituting the approximation (11) for u into the weak form (10), we obtain (dropping the superscript e)

$$\iint_{D^e}\left[\frac{\partial w}{\partial x}\sum_{i=1}^{n}u_i\frac{\partial N_i}{\partial x}+\frac{\partial w}{\partial y}\sum_{i=1}^{n}u_i\frac{\partial N_i}{\partial y}\right]dA=\int_{\Gamma^e} w\,d_n ds+\iint_{D^e} w\,f\,dA \qquad (13)$$

The equation (13) contains n variables $u_1, u_2, \ldots, u_n$. We construct n algebraic equations by choosing $N_1; N_2; ::::: ;N_n$ for w : We can write this n x n set of linear algebraic equations as

$$\sum_{i=1}^{n}A_{ki}u_i = b_k \qquad (14)$$

where

$$A_{ki} = \iint_{D^e}\left[\frac{\partial N_k}{\partial x}\frac{\partial N_i}{\partial x}+\frac{\partial N_k}{\partial y}\frac{\partial N_i}{\partial y}\right]dA$$

$$b_k = \int_{\Gamma^e} N_k\,d_n\,ds+\iint_{D^e} N_k\,f\,dA \qquad (15)$$

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The linear interpolation functions, N_i *(x,y),* for the triangular element are given by

$$N_i(x,y)=\frac{1}{2A}\left(\alpha_i+\beta_i\,x+\gamma_i\,y\right) \quad (i=1,2,3) \qquad (16)$$

Here α_i, β_i and γ_i are constants given by $\alpha_i = x_j y_k - x_k j_j, \beta_i = y_j - y_k$ and $y_i = -(x_j - x_k)$. The evaluation of the element matrices A_{ki} and vectors b_k involves evaluation of double integrals over triangular elements. Most of the time, numerical integration techniques (Gaussian quadrature, area coordinates) are used to compute those integrals.

ASSEMBLY AND IMPOSITION OF BOUNDARY CONDITIONS

To assemble finite element equations we use the continuity of primary variables, *u(x, y),* and the balance of the secondary variables. The continuity of the primary variables at the inter-element nodes guarantees the continuity of the primary variables along the entire inter-element boundary. The assembled equations can be partitioned into:

$$\begin{bmatrix}[A^{11}] & [A^{12}]\\ [A^{21}] & [A^{22}]\end{bmatrix}\begin{Bmatrix}\{U^1\}\\ \{U^2\}\end{Bmatrix}=\begin{Bmatrix}\{B^1\}\\ \{B^2\}\end{Bmatrix} \qquad (17)$$

where $\{U^1\}$ is the column of known primary variables, $\{U^2\}$ is the column of unknown primary variables, $\{B^1\}$ is the column of unknown secondary variables, and $\{B^2\}$ is the column of known secondary variables. Writing (17) as two matrix equations, we get

$$[A^{11}]\{U^1\} + [A^{12}]\{U^2\} = \{B^1\}$$

and

$$[A^{21}]\{U^1\} + [A^{22}]\{U^2\} = \{B^2\}.$$

Now we can find

$$\{U^2\} = [A^{22}]^{-1}\left(\{B^2\} - [A^{21}]\{U^1\}\right)$$

Once $\{U^2\}$ is known, $\{B^1\}$ can be computed.

POSTPROCESSING OF THE SOLUTION

The finite element solution and its derivatives are computed at any point (x, y) in an element e using

$$U^e(x,y) = \sum_{j=1}^{3} u_j^e N_j^e(x,y)$$

$$\frac{\partial U^e}{\partial x} = \sum_{j=1}^{3} u_j^e \frac{\partial N_j^e(x,y)}{\partial x}$$

$$\frac{\partial U^e}{\partial y} = \sum_{j=1}^{3} u_j^e \frac{\partial N_j^e(x,y)}{\partial y} \qquad (18)$$

FLOW AROUND AN ELLIPTIC CYLINDER

The irrotational flow of an inviscid fluid about a elliptical cylinder, placed with its axis perpendicular to the plane of the flow between two long horizontal walls is analyzed using FEM. Due to symmetry, we consider one quadrant, ABCDE, of the whole domain as shown in the figure I. The BVP is written in terms of stream function ψ

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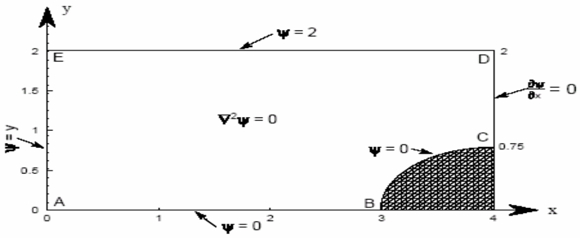

Figure 1. FEM Domain with Boundary Conditions.

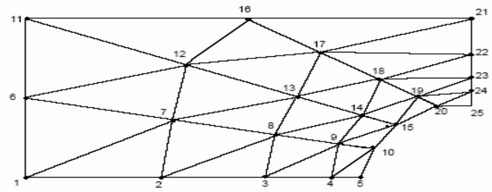

Figure 2. FEM Domain with Nodes.

We use 32 elements and 25 nodes to discretize this domain as shown in figure 2 and the finite element results at the nodes are presented in table 1.

Table I
FEM Solutions at Various Nodes

Node #	x-y coordinate	FEM Solution
7	(1.392, 0.784)	0.759
8	(2.398, 0.628)	0.514
9	(3.017, 0.532)	0.219
12	(1.500, 1.421)	1.401
13	(2.583, 1.003)	0.853
14	(3.250, 0.745)	0.325
17	(2.750, 1.454)	1.327
18	(3.291, 1.059)	0.716
19	(3.625, 0.817)	0.222
22	(4.000, 1.464)	1.221
23	(4.000, 1.077)	0.602
24	(4.000, 0.839)	0.174

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