Reproving Minkowski's Inequality as a Tool for Student Research

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Abstract

Many undergraduate curriculums in mathematics now include some form of a research project. However, even talented students often remain at a loss on exactly how to proceed. Instructors are faced with trying to identify an area of student research that is sufficiently complex so that there is little or no existing results already produced and at the same time trying to insure that the student has the tools necessary to proceed as independently as possible. Most students end up relying heavily on their advisors, but there is help available. In his article, "But How Do I *Do* Mathematical Research?" Suzuki [1] suggests that by first categorizing mathematical research students may find it easier to plan and navigate their own research. This paper focuses on the category of proof as a method of research and uses mathematical induction to reprove a discrete case of the Minkowski Inequality.

Introduction

More and more undergraduate curriculums in mathematics are including some form of student research. Suzuki [1] uses the acronym PEACE as a mnemonic device to represent general categories of undergraduate research: Proof, Extensions, Application, Characterization, and Existence. This paper is concerned with the first category Proof which includes reproof as a significant form of mathematical research.

Every mathematic project includes proof, but reproof is often overlooked as a vital method of mathematical research. Case in point, in 1799 the University of Helmstedt granted Gauss a Ph.D. in mathematics for a dissertation that gave a new proof of the Fundamental Theorem of Algebra, a polynomial, $P(z)$, of degree n has n values z_i for which $P(z_i)=0$. Another fine example is the Pythagorean Theorem, let a, b, and c be two sides and the hypotenuse (respectively) of a right triangle, then $a^2+b^2=c^2$. Indeed, *The Pythagorean Proposition* compiled in 1907 [2] contains approximately 365 distinct proofs of Pythagoras' theorem.

Mathematical Induction

Mathematical Induction (MI) is a method of mathematical proof typically used to establish that a given statement is true of all natural numbers n, or otherwise is true of all members of an infinite sequence which will be demonstrated in this paper. The simplest and most common form of MI consists of two steps, the basic step and the inductive step. The basic step shows that the statement holds when $n = 1$ and is followed by the inductive step showing that if the statement holds for $n = k$ for some $k \geq 1$, then the same statement also holds for $n = k + 1$. For example, suppose we wish to prove the statement: $1+2+3+...+n=\frac{n(n+1)}{2}$ is true $\forall n \in N$. It would take a considerable effort to demonstrate this was true just for the first 100 positive integers. An old, and not necessarily true, tale of Gauss describes how his schoolteacher wanted some quiet and subsequently instructed the class to add up the integers from 1 to 100. Gauss, then a young boy, quickly solved the problem by repeatedly pairing off the biggest and smallest numbers, $(1+100)+(2+99)+...+(50+51)$. Note that each pair adds up to 101, and that there are exactly 50 such pairs. So the sum is 5050. This example often serves as a demonstration of using MI to prove the formula $\sum_{k=1}^{n} k=\frac{n(n+1)}{2} \quad \forall n \in N$ in many mathematical books. However, MI is not limited to proofs involving series. MI can be used for demonstrating divisibility, such as $2^{2n-1}+1$ is divisible by 3 $\forall n \in N$, or for proving inequalities such as, $(1+\varepsilon)^n < 1+3^n \varepsilon$ for $0<\varepsilon<1, \forall n \in N$. Regardless of the mathematical nature of the statement, each proof by MI requires the application of the induction hypothesis. This is usually achieved by relating the statement $P(n+1)$ to the statement $P(n)$. However, this does not mean that proof by MI is a purely procedural activity as we will demonstrate in this paper.

Minkowski's Inequality

In mathematics, the triangle inequality states that given any triangle with sides of length *a*, *b* and *c*, then $c \le a+b$. Equivalently, for complex numbers z_1 and z_2, we write, $|z_1+z_2| \le |z_1|+|z_2|$ where $|z|$ represents the length of a complex number z. If we define $|a|=\left(\sum_{k=1}^{n}|a_k|^2\right)^{1/2}$ to be the length of a vector $a=(a_1,a_2,...,a_n) \in C^n$ where C is the set of all complex numbers, and define the addition of two vectors in the usual way, we have a triangular inequality in C^n. The discrete Minkowski's Inequality with $p=2$ is such a triangular inequality. Here, we introduce a popular proof of this Minkowski's Inequality as the following theorem.

Theorem. Let $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ be complex numbers.

Then, $\left(\sum_{j=1}^{n}|a_j+b_j|^2\right)^{1/2} \le \left(\sum_{j=1}^{n}|a_j|^2\right)^{1/2}+\left(\sum_{j=1}^{n}|b_j|^2\right)^{1/2}$.

Proof [3, p. 25**].** Let $A=\sum_{j=1}^{n}|a_j|^2$, $B=\sum_{j=1}^{n}|b_j|^2$, and $C=\sum_{j=1}^{n}a_j\overline{b_j}$.

If $B=0$ then $b_k=0 \quad \forall k$, and the conclusion is trivial.

If $B > 0$ then $\sum_{j=1}^{n} |a_j + b_j|^2 = \sum_{j=1}^{n} (a_j + b_j)\overline{(a_j + b_j)}$

$$= \sum_{j=1}^{n} a_j \overline{a_j} + \sum_{j=1}^{n} a_j \overline{b_j} + \sum_{j=1}^{n} b_j \overline{a_j} + \sum_{j=1}^{n} b_j \overline{b_j}$$
$$= A + C + \overline{C} + B$$
$$= A + 2\operatorname{Re}(C) + B$$
$$\leq A + 2|C| + B$$
$$\leq A + 2\sqrt{A}\sqrt{B} + B$$
$$= \left(\sqrt{A} + \sqrt{B}\right)^2$$

where the second inequality is Cauchy's Inequality (or CBS-Inequality).

Thus, $\left(\sum_{j=1}^{n} |a_j + b_j|^2\right)^{1/2} \leq \left(\sum_{j=1}^{n} |a_j|^2\right)^{1/2} + \left(\sum_{j=1}^{n} |b_j|^2\right)^{1/2}$.

Now, we reprove the Minkowski's Inequality (discrete case $p = 2$) by using MI.

Proof. For $n = 1$ the inequality is merely a restatement of the triangle inequality in C, $|a_1 + b_1| \leq |a_1| + |b_1|$. Now, assume that Minkowski's Inequality is true for $n = N$ for some $N \geq 1$ and use MI to show that

$$\left(\sum_{j=1}^{N+1} |a_j + b_j|^2\right)^{1/2} \leq \left(\sum_{j=1}^{N+1} |a_j|^2\right)^{1/2} + \left(\sum_{j=1}^{N+1} |b_j|^2\right)^{1/2}.$$

We define that $\sum_{j=1}^{0} |z_j| = 0$ for our convenience.

Begin with LHS,

$$\left(\sum_{j=1}^{N+1} |a_j + b_j|^2\right)^{1/2} = \left(\sum_{j=1}^{N-1} |a_j + b_j|^2 + |a_N + b_N|^2 + |a_{N+1} + b_{N+1}|^2\right)^{1/2}$$

$$= \left(\sum_{j=1}^{N-1} |a_j + b_j|^2 + (a_N + b_N)\overline{(a_N + b_N)} + (a_{N+1} + b_{N+1})\overline{(a_{N+1} + b_{N+1})}\right)^{1/2}$$

$$= \left(\sum_{j=1}^{N-1} |a_j + b_j|^2 + |a_N|^2 + |b_N|^2 + |a_{N+1}|^2 + |b_{N+1}|^2 + 2\operatorname{Re}\left(a_N \overline{b_N}\right) + 2\operatorname{Re}\left(a_{N+1}\overline{b_{N+1}}\right)\right)^{1/2}.$$

Notice that $\left(|a_N \overline{b_N}| + |a_{N+1}\overline{b_{N+1}}|\right)^2 \leq \left(|a_N|^2 + |a_{N+1}|^2\right)\left(|b_N|^2 + |b_{N+1}|^2\right)$, we have

$$2\,\mathrm{Re}(a_N\overline{b_N}) + 2\,\mathrm{Re}(a_{N+1}\overline{b_{N+1}}) \le 2\sqrt{\left(|a_N|^2+|a_{N+1}|^2\right)\left(|b_N|^2+|b_{N+1}|^2\right)},$$

and the LHS is less than or equal to

$$\left(\sum_{j=1}^{N-1}\left|a_j+b_j\right|^2 + |a_N|^2+|b_N|^2+|a_{N+1}|^2+|b_{N+1}|^2+2\sqrt{\left(|a_N|^2+|a_{N+1}|^2\right)\left(|b_N|^2+|b_{N+1}|^2\right)}\right)^{1/2}.$$

Defining $\alpha=\sqrt{\left(|a_N|^2+|a_{N+1}|^2\right)}$ and $\beta=\sqrt{\left(|b_N|^2+|b_{N+1}|^2\right)}$ and applying the induction hypothesis, we have,

$$\begin{aligned}\left(\sum_{j=1}^{N+1}\left|a_j+b_j\right|^2\right)^{1/2} &\le \left(\sum_{j=1}^{N-1}\left|a_j+b_j\right|^2+|\alpha+\beta|^2\right)^{1/2}\\ &\le \left(\sum_{j=1}^{N-1}\left|a_j\right|^2+|\alpha|^2\right)^{1/2}+\left(\sum_{j=1}^{N-1}\left|b_j\right|^2+|\beta|^2\right)^{1/2}\\ &= \left(\sum_{j=1}^{N+1}\left|a_j\right|^2\right)^{1/2}+\left(\sum_{j=1}^{N+1}\left|b_j\right|^2\right).\end{aligned}$$

We have completed the MI and hence,

$$\left(\sum_{j=1}^{n}\left|a_j+b_j\right|^2\right)^{1/2} \le \left(\sum_{j=1}^{n}\left|a_j\right|^2\right)^{1/2}+\left(\sum_{j=1}^{n}\left|b_j\right|^2\right)^{1/2} \text{ is true } \forall n\in N.$$

Conclusion

There are many examples of mathematical proofs introduced in undergraduate courses that can be reproven through a variety of different processes. Here we demonstrated reproof using MI, but in this proof the statement $P(n-1)$ is cleverly related back to the statement $P(n+1)$. This reproof of Minkowski's Inequality is mathematically difficult enough to be a significant learning experience for the student, but at the same time uses only elementary properties of complex numbers, summations, and algebra, and the method of MI, thereby encouraging the student to work relatively independently.

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References

1. J. Suzuki, *But How do I Do Mathematical Research?*, *Focus* 24 (2004) 10.
2. E. S. Loomis, *The Pythagorean Proposition*, NCTM, 1968.

3. K. R. Stromberg, *An Introduction to Classical Real Analysis*, Wadsworth, Inc., 1981.