An Algebraic Characterization of ER-Reduced Dependence Alphabets

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Abstract

It is well known that the endomorphism monoid of an arbitrary dependence alphabet is regular. In this short note, we will give an algebraic characterization of ER-reduced dependence alphabets, i.e., dependence alphabets are ER-reduced if and only if their endomorphism monoids are groups.

Introduction

During the past years, monoids of endomorphisms of graphs were investigated (Knauer, 1990; Knauer & Nieporte, 1989). In (Knauer & Nieporte, 1989), the authors proved that such a monoid is regular for a finite undirected graph without multiple edges. Roughly speaking, a *dependence alphabet* is a finite undirected graph with a loop on each node. The importance of this kind of graphs is their intrinsic relation with the theory of traces. The latter has led to interesting problems and significant results in quite diverse areas that include formal language theory, combinatorics, graph theory, algebra, logic, and the theory of concurrent systems. For details, please refer to (Diekert & Rozenberg, 1995) and its extensive bibliography. In (Ehrenfaucht & Rozenberg, 1987), the authors have studied dependence alphabets and have obtained some important results. The objective of this paper is to investigate the structures of the endomorphism monoids of dependence alphabets.

Preliminaries

In this section we briefly sketch the notation and preliminaries we will use throughout this paper.

For an arbitrary set X, $\omega(X)$ denotes its cardinality and $\iota_X = \{(x,x) \mid x \in X\}$; $\varnothing$ denotes the empty set. For sets X and Y, $X \setminus Y$ denotes their difference. If Σ is a partition of a set X and $x \in X$, then $[x]_\Sigma$ denotes the class of Σ containing x. Let α be a mapping from set X to set Y, then $\mathrm{Im}(\alpha)$ denotes the image of α.

A *directed graph* is specified in the form $G = (V, E)$ where, $V \neq \varnothing$ is a finite set of *nodes* of G and $E \subseteq V \times V$ is the set of *edges* of the graph. We also use $V(G)$ and $E(G)$ to denote V and E, respectively. Ordered pairs of nodes denote edges of a directed graph G. A directed graph G is called *symmetric* if and only if $(v,u) \in E(G)$ whenever $(u,v) \in E(G)$ for all

$u, v \in V(G)$. A symmetric graph is also called an *undirected* graph. We say that a graph G is *reflexive* if and only if $\iota_{V(G)} \in E(G)$.

Definition 2.1 A dependence alphabet is a symmetric and reflexive graph.

Note that, in any dependence alphabet G, an edge is denoted by the unordered pair $\{u, v\}$ whenever $u, v \in G$.

Let $G = (V, E)$ be a dependence alphabet. The subgraph induced by a nonempty subset of nodes $U \subseteq V$ is the graph $G(U) = (U, E(G) \cap U \times U)$. Clearly, $G(U)$ is still a dependence alphabet. A dependence alphabet $G = (V(G), E(G))$ is *discrete* if $E(G) = \iota_{V(G)}$. G is *complete* if $E(G) = V \times V$. A complete subgraph of G is called a *clique*.

Definition 2.2 A nonempty subset $U \subseteq V(G)$ in a dependence alphabet G is called a *clan* if for all $u, v \in U$ and each $w \in V(G) \setminus U$, $\{w, u\} \in E(G)$ if and only if $\{w, v\} \in E(G)$.

Let G and H be dependence alphabets. A mapping $\alpha : V(G) \to V(H)$ is a *homomorphism* from G to H if, $\{u, v\} \in E(G)$ if and only if $\{\alpha(u), \alpha(v)\} \in E(H)$ for all $u, v \in V(G)$. α is an *isomorphism* if and only if α is a one-one correspondence. We use $Hom(G, H)$ and $Isom(G, H)$ to denote the set of all homomorphisms and all isomorphisms from G to (onto) H, respectively. We denote $End(G) = Hom(G, G)$ and $Aut(G) = Isom(G, G)$. Clearly, if $\alpha \in Isom(G, H)$ then α is a bijection and its inverse α^{-1} is an isomorphism from H to G. It is well known, for any dependence alphabet, $End(G)$ is a monoid while $Aut(G)$ is a group with respect to the composition of mappings.

Let G be a dependence alphabet. The similarity relation of G, denoted by sim_σ, is the binary relation on $V(G)$ such that for all $u, v \in V(G)$, $u \ sim_\sigma \ v$ if and only if $\{u, v\}$ is a clan of G. By [(Ehrenfaucht & Rozenberg, 1987); Theorem 1.1] sim_σ is an equivalence relation on $V(G)$. The set of all equivalence classes of sim_σ is denoted by Σ. It is not difficult to show that $G[v]_\Sigma$ is either a discrete subgraph or a clique.

Let G be a dependence alphabet and let $\widetilde{\Sigma}$ be a partition of $V(G)$ and $\widetilde{G}=(\widetilde{\Sigma},\widetilde{E})$ satisfies the conditions: (1) If $[v]_{\Sigma}$ is a clique then $[v]_{\widetilde{\Sigma}}=[v]_{\Sigma}$, (2) If $[v]_{\Sigma}$ is a discrete subgraph then $[v]_{\widetilde{\Sigma}}=\{v\}$, and (3) For all $\widetilde{u},\widetilde{v}\in\widetilde{\Sigma}$, $\{\widetilde{u},\widetilde{v}\}\in\widetilde{E}$ if and only if there exists an $x\in\widetilde{u}$ and $y\in\widetilde{v}$ such that $\{x,y\}\in E(G)$. Then, $\widetilde{G}$ is a dependence alphabet. Therefore, the following concept is well defined:

Definition 2.3 A dependence alphabet G is called *Ehrenfeucht-Rozenberg Reduced*, or *ER-Reduced*, if and only if $G=\widetilde{G}$.

The Structure of $End(G)$ for ER-Reduced Dependence Alphabets

The key result of this section provides an algebraic characterization of ER-reduced dependence alphabet by means of its homomorphism monoid, namely, a dependence alphabet is ER-reduced if and only if G is unretractive, i.e., $End(G)=Aut(G)$. To prove this main result, we need several lemmas.

Lemma 3.1 [(Ehrenfaucht & Rozenberg, 1987); Lemma 1.3] Let G be a dependence alphabet and let U_1 and U_2 be disjoint clans of G. Then, for all $u_1,v_1\in U_1$ and all $u_2,v_2\in U_2$, $\{u_1,v_2\}\in E(G)$ if and only if $\{u_2,v_1\}\in E(G)$.

A clan U of G is called *complete* if $G(U)$ is a clique.

Lemma 3.2 Let G and H be dependence alphabets and let $\alpha\in Hom(G,H)$, then $\alpha^{-1}(K)$ is a complete clan of G for each complete clan K of H.

Proof We first prove that $\alpha^{-1}(K)$ is a clan of G. Indeed, if $u,v\in\alpha^{-1}(K)$ and $w\in V(G)\setminus\alpha^{-1}(K)$, then $\alpha(u),\alpha(v)\in V(H)$ and $\alpha(w)\in V(H)\setminus K$. We deduce that $\{w,u\}\in E(G)\Leftrightarrow\{\alpha(w),\alpha(u)\}\in E(H)\Leftrightarrow\{\alpha(w),\alpha(v)\}\in E(H)\Leftrightarrow$ $\{w,v\}\in E(G)$. That is to say, $\alpha^{-1}(K)$ is a clan. Next, we show that $G(\alpha^{-1}(K))$ is a clique. In fact, for all $u,v\in\alpha^{-1}(K)$, we have $\alpha(u),\alpha(v)\in K$. Since K is a clique of H, we know that $H(K)$ is complete. This means that $\{u,v\} \in E(H(K))$, or $\{\alpha(u), \alpha(v)\} \in E(H)$ which, in turn, implies that $\{u,v\} \in E(G)$ since, α is a homomorphism. Therefore, $\{u,v\} \in E(G(\alpha^{-1}(K)))$.

Because each singleton in a dependence alphabet is a complete clan, we have the following

Corollary 3.3 Let G and H be dependence alphabets and let α be an homomorphism from G to H, then $\alpha^{-1}(v)$ is a complete clan of G for each $v \in V(H)$.

Let G and H be dependence alphabets and $\alpha \in Hom(G,H)$, we denote

$$\Xi_\alpha = \{\alpha^{-1}(k) \mid k \in V(H) : \alpha^{-1}(k) \neq \emptyset\}.$$

Clearly, Ξ_α is a partition of $V(G)$ to complete clans. This situation holds also for the other way around:

Lemma 3.4 Let G be a dependence alphabet and Ξ be a partition of $V(G)$ to complete clans. Defining a mapping $\alpha : V(G) \to \Xi$ such that $\alpha(u) = [u]_\Xi$ for all $u \in V(G)$. Then $\alpha \in Hom(G, \tilde{G})$ is an epimorphism, where, $\tilde{G} = (\Xi, \tilde{E})$ and $\tilde{E}$ is specified in the manner that for all $\xi_1, \xi_2 \in \Xi$, $\{\xi_1, \xi_2\} \in \tilde{E}$ if and only if there exist nodes u and v of G such that $u \in \xi_1, v \in \xi_2$ and $\{u,v\} \in E(G)$.

Remark 3.5 According to *Lemma 3.2*, if $\Lambda \subseteq \Xi$ is a complete clan of $\tilde{G}$ then

$$K = \bigcup_{[u]_\Xi \in \Lambda} [u]_\Xi$$

is a complete clan of G. Therefore, Examples 2.2 and 2.3 in (Ehrenfaucht & Rozenberg 1987) do not stand up to a closer examination.

By *Definition 2.3*, we know that $\tilde{\Sigma}$ is a partition of $V(G)$ into complete clans. Moreover, the following *maximal property* holds:

Lemma 3.6 [(Ehrenfaucht & Rozenberg, 1987); Lemma 2.7] Let G be a dependence alphabet. Then $\tilde{\Sigma}$ is a maximal partition of G into complete clans. That is to say, if Π is an arbitrary partition of $V(G)$ into complete clans, then Π is a refinement of $\tilde{\Sigma}$.

Lemma 3.7 Let G be a dependence alphabet, then the mapping $\beta : \tilde{\Sigma} \to V(G)$ defined by $\beta([v_i]_{\tilde{\Sigma}}) = v_i$ is a monomorphism from $\tilde{G}$ to G, where, $\{v_i\}$ is an arbitrary but fixed set of representatives of the partition $\tilde{\Sigma}$.

Proof By definition of $\tilde{\Sigma}$, $\{[v_i]_{\tilde{\Sigma}}, [v_j]_{\tilde{\Sigma}}\} \in \tilde{E}$ if and only if there exist $u \in [v_i]_{\tilde{\Sigma}}$ and $v \in [v_j]_{\tilde{\Sigma}}$ such that $\{u, v\} \in E(G)$. If $i \neq j$, by assumption, we have $[v_i]_{\tilde{\Sigma}} \neq [v_j]_{\tilde{\Sigma}}$. By Lemma 3.1, we have $\{u, v\} \in E(G)$ if and only if $\{v_i, v_j\} \in E(G)$. This is equivalent to, $\{[v_i]_{\tilde{\Sigma}}, [v_j]_{\tilde{\Sigma}}\} \in \tilde{E}$ if and only if $\{v_i, v_j\} \in E(G)$. Therefore, β is a homomorphism. According to Lemmas 3.2 and 3.6, β is one-to-one.

Now, we are ready to prove our main result.

Theorem 3.8 A dependence alphabet G is ER-reduced if and only if $End(G) = Aut(G)$.

Proof If G is ER-reduced, i.e., $\tilde{\Sigma} = V(G)$, then $End(G) = Aut(G)$. Assume that $End(G) \neq Aut(G)$ then there exists a $\alpha \in End(G) \setminus Aut(G)$. By the finiteness of $V(G)$, α must not be one-to-one. Hence, there exists $v \in V(G)$ such that $\omega(\alpha^{-1}) > 1$. According to Corollary 3.3, $\alpha^{-1}(v)$ is a complete clan. Thus, $\{\alpha^{-1}(v)\} \bigcup \{\{u\} \mid u \in V(G) \setminus \alpha^{-1}(v)\}$ is a partition of $V(G)$ into complete clans, which contradicts to the maximal property of Σ.

On the other hand, let G be an arbitrary dependence alphabet that satisfies the condition $End(G) = Aut(G)$, then G is ER-reduced. In fact, assume that $G \neq \tilde{G}$, then based on Lemmas 3.4 and 3.7, there is an epimorphism

$\alpha : G \to \tilde{G}$ and a monomorphism $\beta : \tilde{G} \to G.$ Clearly, the composition $\beta \circ \alpha \in End(G) \setminus Aut(G),$ a contradiction.

Examples 3.9 (1) Each discrete dependence alphabet G is ER-reduced. As a matter of fact, $End(G) = Aut(G) = S_n,$ the symmetric group of degree $n = \omega(V(G)).$

(2) A complete dependence alphabet G is ER-reduced only if it is a singleton. Indeed, $End(G) = T_n,$ the transformation monoid of degree $n = \omega(V(G)).$

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