Dispersion Relation and Wave Loads on a Vertical Cylinder in Water due to First Order Diffraction

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Abstract

Here we consider the first order wave diffraction by a cylindrical structure. The cylindrical structure is circular, vertical, surface piercing in water of finite depth. First we state the boundary value problem in terms of velocity potential function. The formulation of the wave-structure interaction is based on the assumption of a homogeneous, ideal, incompressible, and inviscid fluid. Inclusion of irrotationality allows us to introduce the velocity potential function which satisfies the Laplace equation in the fluid domain. The nonlinearity for the wave-structure interaction problem arises from the free surface boundary conditions. We use the separation of variables method to solve the first order problem by writing the total velocity potential as the sum of the incident velocity potential and the scattered velocity potential. From the combined free surface condition, we derive the dispersion relation. After deriving the pressure using Bernoulli's equation, we obtain the analytical expression for the first order force on the cylinder by integrating the pressure over the wetted surface. Numerical results for the dispersion relation and wave loads for various depths to radius ratios are presented.

Introduction

The computation of the water wave forces on offshore structures is one of the main interests in designing safe offshore structures. The structure may be fixed or floating as semi-merged structure in sea. There is a large number of structures which are composed of tubular members like circular cylinders. When the structure spans a significant amount of wavelength, the incident waves undergo scattering or diffraction. Diffraction of waves needs to be considered while evaluating the wave forces.

Dean and Dalrymple [2] presented a review of potential flow hydrodynamics. Solutions for standing and progressive small amplitude water waves provide the basis for application to numerous problems of engineering interest. They discussed the formulation of the linear water ware theory and development of the simplest two-dimensional solution for standing and progressive waves. Debnath [3] discussed theoretical studies of nonlinear water waves over the last few decades. His work is primarily devoted to the mathematical theory of nonlinear water waves with applications. He studied the theory of nonlinear shallow water waves and solitons, with emphasis on methods and solutions of several evolution equations that are originated in the theory of water waves. Johnson [4] describes the mathematical ideas and techniques that are directly relevant to water wave theory. Beginning with the introduction of the appropriate equations of fluid mechanics, together with the relevant boundary conditions, the ideas of nondimensionalisation, scaling and asymptotic expansions are briefly explored. Rahman [6] presented an introduction to the mathematical and physical aspects of the theory of water waves. He discussed the wave theory of Airy, nonlinear wave theory of Stokes, tidal dynamics in shallow water. He mentioned about the dynamics of floating offshore structures.

In the present work, we present the boundary value problem for a fixed cylinder in water of uniform depth. The velocity potential function satisfies the Laplace equation. We use the separation of variables method to solve the first order problem by writing the total velocity potential as the sum of the incident velocity potential and the scattered velocity potential. The solution is obtained in terms of Bessel's functions. From the combined free surface condition, we derive the dispersion relation. After deriving the pressure using Bernoulli's equation, we obtain the analytical expression for the first order force on the cylinder by integrating the pressure over the wetted surface. Numerical results for the dispersion relation and wave loads for various depth to radius ratios are presented.

Velocity Potential Function

Here we consider a fixed vertical cylinder in water of finite uniform depth. The cylinder extends from sea bed (z = -h) to the free surface $z = \eta(x, y, t)$. Water depth is h. $\eta(x, y, t)$ is the free surface elevation function and r is the radius of the cylinder. Incident wave is propagating along positive x-direction. The incoming wave incident upon the surface of the cylinder undergoes a diffraction or scattering. To evaluate the wave loads in the cylinder we need to consider the effect of the incident wave and the diffracted wave.

The cylindrical coordinate system (r, θ, z) with z vertically upwards from the still water level (SWL), r measured radially from the from z-axis and θ from the positive x-axis is used. For Cartesian coordinates (x, y, z), xy-plane represents the still water level (SWL) and z-axis positive upward from the SWL. Cartesian and Cylindrical coordinates are related by $x = r\cos\theta$, $x = r\sin\theta$, z=z. The formulation is based on the assumptions of ideal, incompressible and inviscid fluid. We assume that sea floor is flat and horizontal, and situated at z = -h.

The equation of continuity for a fluid with velocity, v and density, ρ is given by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0 \qquad (1)$$

For incompressible fluid, the continuity equation is

$$\nabla \cdot v = 0 \tag{2}$$

For an incompressible and inviscid fluid with irrotational motion, we can introduce a velocity $\phi(r,\theta,z,t)$ such that

$$v = \nabla \phi(r,\theta,z,t) \tag{3}$$

Equations (2) and (3) yield that the velocity potential satisfies the Laplace equation in the fluid domain, i.e.,

$$\nabla^2 \phi(r,\theta,z,t) = 0 \tag{4}$$

We will assume that $\phi(r,\theta,z,t)$ is time harmonic.

The horizontal force components F_x, F_y along x, y directions are given by

$$F_x = \int_{\theta=0}^{2\pi} \int_{z=-h}^{\eta} P(a,\theta,z,t)(-\cos\theta)\, a\, dz\, d\theta \tag{5}$$

$$F_y = \int_{\theta=0}^{2\pi} \int_{z=-h}^{\eta} P(a,\theta,z,t)(-\sin\theta)\, a\, dz\, d\theta \tag{6}$$

respectively. Here is the $P(a,\theta,z,t)$ is the pressure on the curved surface of the cylinder which can be computed from the velocity potential using Bernoulli's equation. $\eta(x,y,t)$ is the free surface elevation function.

Boundary Value Problem in terms of Velocity Potential and Free Surface Elevation Functions

The boundary value problem in terms of that $\phi(r,\theta,z,t)$ and $\eta(x,y,t)$ is can be expressed as

Governing Equation:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{7}$$

Dynamic free surface boundary condition at the free surface $z = \eta(x,y,t)$:

$$\frac{\partial \phi}{\partial t} + g\eta + \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right] = 0 \tag{8}$$

Kinematic free surface boundary condition at the free surface $z = \eta(x, y, t)$:

$$\frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \eta}{\partial y} - \frac{\partial \phi}{\partial z} = 0 \qquad (9)$$

Since the cylinder is fixed, the normal velocity is zero, so the body surface boundary condition on the curved surface:

$$\frac{\partial \phi}{\partial n} = 0, \qquad r = a \qquad (10)$$

where n is the outward normal. Assuming that sea floor is flat and horizontal, the bottom boundary condition at z = -h can be expressed

$$\frac{\partial \phi}{\partial z} = 0, \qquad z = -h \qquad (11)$$

The velocity potential and the surface elevation can be written in terms of Stokes expansion as

$$\phi(x, y, z, t) = \varepsilon \varphi_1(x, y, z, t) + \varepsilon^2 \phi_2(x, y, z, t) + O(\varepsilon^3)$$

and

$$\eta(x, y, z, t) = \varepsilon \eta_1(x, y, z, t) + \varepsilon^2 \eta_2(x, y, z, t) + O(\varepsilon^3)$$

Here $\varepsilon \quad (= kA)$ is the dimensionless small parameter where k the wavenumber, A wave amplitude. Sub-index 1 is used to represent the first expansion term corresponding to a linear approximation and sub-index 2 is used for the second order approximation.

At the free surface, we have $z = \eta(x, y, t)$, so $\phi(x, y, z, t) = \phi(x, y, \eta, t)$. Expanding by Taylor's theorem about z = 0, we have the modified velocity potential at the free surface is

$$\phi(x, y, \eta, t) = \varepsilon \varphi_1(x, y, 0, t) + \varepsilon^2 \left[\phi_2(x, y, 0, t) + \eta_1 \left(\frac{\partial \phi_1}{\partial z} \right)_{z=0} \right] + O(\varepsilon^3)$$

Dynamic free surface boundary condition now is given by

$$\varepsilon \left(\frac{\partial \phi_1}{\partial t} + g\eta_1 \right) + \varepsilon^2 \left[\frac{\partial \phi_2}{\partial t} + g\eta_2 + \eta_1 \frac{\partial^2 \phi_1}{\partial t \partial z} \right] + \frac{\varepsilon^2}{2} \left[\left(\frac{\partial \phi_1}{\partial x} \right)^2 + \left(\frac{\partial \phi_1}{\partial y} \right)^2 + \left(\frac{\partial \phi_1}{\partial z} \right)^2 \right] + O\left(\phi^3\right) = 0$$

Kinematic free surface boundary condition now can be written as

$$\varepsilon\left(\frac{\partial\eta_1}{\partial t}-\frac{\partial\phi_1}{\partial z}\right)+\varepsilon^2\left[\frac{\partial\eta_2}{\partial t}+\frac{\partial\phi_1}{\partial x}\frac{\partial\eta_1}{\partial x}\right.$$
$$\left.+\frac{\partial\phi_1}{\partial y}\frac{\partial\eta_1}{\partial y}-\frac{\partial\phi_2}{\partial z}-\eta_1\frac{\partial^2\phi_1}{\partial z^2}\right]+O(\phi^3)=0$$

Now comparing the coefficients of ε we can separate the free surface boundary conditions for various order. For the first order, at z = 0, the dynamic free surface boundary condition is

$$\frac{\partial\phi_1}{\partial t}+g\eta_1=0 \qquad (12)$$

and the kinematic condition becomes

$$\frac{\partial\eta_1}{\partial t}-\frac{\partial\phi_1}{\partial z}=0 \qquad (13)$$

Eliminating the first order free surface elevation function from the equations (12) and (13), the combined free surface condition can be expressed

$$\frac{\partial^2\phi_1}{\partial t^2}+g\frac{\partial\phi_1}{\partial z}=0, \quad z=0 \qquad (14)$$

The pressure P is determined from Bernoulli's equation

$$\frac{P}{\rho}+gz+\frac{\partial\phi}{\partial t}+\frac{1}{2}(\nabla\phi)^2=0$$

where

$$(\nabla\phi)^2=\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2=\left(\frac{\partial\phi}{\partial r}\right)^2+\left(\frac{\partial\phi}{r\partial\theta}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2.$$

Substituting series expansion for the velocity potential, we have

$$P=-\rho gz-\varepsilon\rho\frac{\partial\phi_1}{\partial t}-\varepsilon^2\rho\left[\frac{\partial\phi_2}{\partial t}+\frac{1}{2}(\nabla\phi_1)^2\right]+O(\varepsilon^3) \qquad (15)$$

Since the incident wave is propagating in x-direction, y-component of the horizontal force vanishes, and the x-component is given by

$$F_x=-\int_{\theta=0}^{2\pi}\int_{z=-h}^{\eta}P(a,\theta,z,t)\cos\theta\, a\, dz\, d\theta \qquad (16)$$

Writing the z-integral as the sum of $\int_{-h}^{0}+\int_{0}^{\eta}$ we obtain

$$F_x = \rho a \int_0^{2\pi} \left[\int_{-h}^{0} \left\{ gz + \varepsilon \frac{\partial \phi_1}{\partial t} + \varepsilon^2 \left(\frac{\partial \phi_2}{\partial t} + \frac{1}{2} (\nabla \phi_1)^2 \right) \right\}_{r=a} dz \right.$$
$$\left. + \int_0^{\varepsilon\pi_1 + \varepsilon^2 \eta_2} \left\{ gz + \varepsilon \frac{\partial \phi_1}{\partial t} + \varepsilon^2 \left(\frac{\partial \phi_2}{\partial t} + \frac{1}{2} (\nabla \phi_1)^2 \right) \right\}_{r=a} dz \right] \cos\theta \, d\theta \qquad (17)$$

Now we write

$$F_x = \varepsilon F_{x_1} + \varepsilon^2 F_{x_2} + \varepsilon^3 F_{x_3} + \ldots\ldots$$

where F_{x_i} represents the i-th order contribution.

Thus the first order force component is given by

$$F_{x_1} = \rho\, a \int_0^{2\pi} \left\{ \int_{-h}^{0} \left(\frac{\partial \phi_1}{\partial t} \right)_{r=a} dz \right\} \cos\theta \, d\theta \qquad (18)$$

First Order Diffraction Problem

The boundary value problem for first order potential ϕ_{1} becomes Governing Equation:

$$\frac{\partial^2 \phi_1}{\partial r^2} + \frac{1}{r}\frac{\partial \phi_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi_1}{\partial \theta^2} + \frac{\partial^2 \phi_1}{\partial z^2} = 0 \qquad (19)$$

Combined free surface boundary condition:

$$\frac{\partial^2 \phi_1}{\partial t^2} + g \frac{\partial \phi_1}{\partial z} = 0, \qquad z = 0 \qquad (20)$$

Body surface boundary condition:

$$\frac{\partial \phi_1}{\partial r} = 0, \qquad r = a \qquad (21)$$

Bottom boundary condition:

$$\frac{\partial \phi_1}{\partial z} = 0, \qquad z = -h \qquad (22)$$

We assume the $\phi(r, \theta, z, t)$ is time harmonic, i.e.,

$$\phi_1(r, \theta, z, t) = \mathrm{Re}\left[\Phi_1(r, \theta, z)\, e^{-i\sigma t} \right] \qquad (23)$$

where $\Phi_1(r, \theta.z)$ is the complex velocity potential and σ is the frequency of the wave. Now we solve the linear boundary value problem by decomposing the complex velocity potential into incident potential $\Phi^{(I)}{}_1(r,\theta.z)$ and scattered potential $\Phi^{(S)}{}_1(r,\theta.z)$ as follows

$$\Phi_1(r,\theta,z) = \Phi^{(I)}{}_1(r,\theta,z) + \Phi^{(S)}{}_1(r,\theta,z) \qquad (24)$$

The complex incident velocity potential can be obtained as

$$\Phi_1{}^{(I)} = \frac{gA}{\sigma}\frac{\cosh k(z+h)}{\cosh kh}\sum_{m=0}^{\infty}\beta_m\, i^m\, J_m(kr)\cos m\theta$$

and the complex scattered velocity potential can be derived as

$$\Phi_1{}^{(S)} = -\frac{gA}{\sigma}\frac{\cosh k(z+h)}{\cosh kh}\sum_{m=0}^{\infty}\beta_m\, i^m \frac{J'_m(ka)}{H^{(1)'}{}_m(ka)} H^{(1)}{}_m(kr)\cos m\theta$$

Here $\beta_0 = 1$, $\beta_m = 2, \quad m \geq 1$; $J_m(kr)$ and $H^{(1)}{}_m(kr)$ $(= J_m(kr) + iY_m(kr))$ are Bessel and Hankel functions of first kind of order m respectively [1, 5]. We use the separation of variables method to obtain these solutions by writing $\Phi_1(r,\theta,z) = R(r)\,\Theta(\theta)\,Z(z)$. This yields Bessel equation, for the incident potential we obtain the solution in terms of Bessel function of first kind and for the scattered potential we obtain the solution in terms of Hankel function of first kind. Scattered potential satisfies the radiation condition. Thus the first order velocity potential can be written as

$$\phi_1(r,\theta,z,t) = \frac{gA}{\sigma}\frac{\cosh k(z+h)}{\cosh kh}\,\mathrm{Re}\left[\sum_{m=0}^{\infty}\beta_m\, i^m \left\{J_m(kr) - \frac{J'_m(ka)}{H^{(1)'}{}_m(ka)} H^{(1)}{}_m(kr)\right\} e^{-i\sigma t}\cos m\theta\right]$$

(25)

Dispersion Relation for the First Order Theory

Combined free surface condition for the first order complex velocity potential is given by

$$g\frac{\partial\Phi_1(r,\theta,z)}{\partial z} - \sigma^2\Phi_1(r,\theta,z) = 0, \qquad on \quad z = 0$$

Now, on z = 0, we have $\frac{\partial \Phi_1}{\partial z} = (k \tanh kh)\, \Phi_1.$ From the combined free surface

condition, we can write $g(k \tanh kh)\, \Phi_1 . - \sigma^2 \Phi_1 = 0$. Thus the dispersion relation can be written as

$$\sigma^2 = g\, k \tanh kh \qquad (26)$$

which describes a relation between the wavenumber, k and the frequency, σ.
To obtain computational results, we consider non-dimensional parameters ka and kh = (ka.h/a) so that we have

$$\frac{\sigma^2 a}{g} = ka \tanh\left(ka \frac{h}{a} \right) \qquad (27)$$

Now we present the numerical results of the dispersion relation for various depth to radius ratios. We consider the following ratios: h/a = 1, h/a = 2, h/a = 5. The following table describes the values of ka for which

$$\frac{\sigma^2 a}{g} - ka \tanh\left(ka \frac{h}{a} \right) = 0.$$

First Order Force Component

First order complex velocity potential is

$$\Phi_1(r,\theta,z,t) = \frac{gA}{\sigma} \frac{\cosh k(z+h)}{\cosh kh} \left[\sum_{m=0}^{\infty} \beta_m\, i^m \left\{ J_m(kr) - \frac{J'_m(ka)}{H^{(1)^1}{}_m(ka)} H^{(1)}{}_m(kr) \right\} \cos m\theta \right] \qquad (28)$$

Since

$$\int_0^{2\pi} \cos m\theta \cos n\theta \, d\theta \quad = 0 \qquad m \neq n$$
$$= \pi \qquad m = n$$

and $\cos\theta$ appears in the force computation, we need to consider only terms involving $\cos\theta$ in the expression of $\Phi_1(r,\theta.z)$. So for m = 1 and r = a, we obtain

$$\frac{\partial \phi_1}{\partial t} = \frac{gA}{\sigma}\frac{\cosh k(z+h)}{\cosh kh}\mathrm{Re}\left[2i(-i\sigma)e^{-i\sigma t}\left\{J_1(ka) - \frac{J'_1(ka)}{H^{(1)'}_1(ka)}H^{(1)}_1(ka)\right\}\right]\cos\theta$$

$$= \frac{gA}{\sigma}\frac{\cosh k(z+h)}{\cosh kh}\mathrm{Re}\left\{\frac{2i(-i\sigma)e^{-i\sigma t}(2i)}{\pi ka\, H^{(1)'}_1(ka)}\right\}\cos\theta$$

$$= \frac{4gA}{\pi ka}\frac{\cosh k(z+h)}{\cosh kh}\mathrm{Re}\left\{\frac{ie^{-i\sigma t}}{H^{(1)'}_1(ka)}\right\}\cos\theta$$

Thus the first order force is

$$F_{x_1} = \rho a\int_0^{2\pi}\left\{\int_{-h}^{0}\left(\frac{\partial\phi_1}{\partial t}\right)_{r=a} dz\right\}\cos\theta\, d\theta$$

$$= \frac{4\rho gA}{\pi k}\int_0^{2\pi}\int_{-h}^{0}\frac{\cosh k(z+h)}{\cosh kh}dz\cos^2\theta\ d\theta$$

$$\times\mathrm{Re}\left\{i\frac{\{J'_1\cos\sigma t - Y'_1\sin\sigma t\} - i\{J'_1\sin\sigma t + Y'_1\cos\sigma t\}}{J'^2_1 + Y'^2_1}\right\}$$

$$= \frac{4\rho gA}{\pi a}\frac{\frac{Y'_1}{\sqrt{J'^2_1+Y'^2_1}}\cos\sigma t + \frac{J'_1}{\sqrt{J'^2_1+Y'^2_1}}\sin\sigma t}{\sqrt{J'^2_1+Y'^2_1}}\frac{\tanh kh}{k}$$

Defining

$$\cos\alpha = \frac{Y'_1(ka)}{\sqrt{J'^2_1(ka)+Y'^2_1(ka)}}$$ and

$$\sin\alpha = \frac{J'_1(ka)}{\sqrt{J'^2_1(ka)+Y'^2_1(ka)}}$$

we can write

$$F_{x_1} = \frac{4\rho gA}{k^2}\frac{\cos(\sigma t-\alpha)}{\sqrt{J'^2_1(ka)+Y'^2_1(ka)}}\tanh kh$$

(29)

where

$$\alpha = \tan^{-1}\left\{\frac{J'_1(ka)}{Y'_1(ka)}\right\}$$

Non-dimensional component is

$$\frac{F_{x_1}}{\rho g a^2 A} = \frac{4}{(ka)^2}\frac{\cos(\sigma t - \alpha)}{\sqrt{J'^{\,2}_1(ka) + Y'^{\,2}_1(ka)}}\tanh kh$$

(30)

Thus the maximum non-dimensional first order force is given by

$$\frac{F_{x_1}^{\max}}{\rho g a^2 A} = \frac{4}{(ka)^2}\frac{\tanh kh}{\sqrt{J'^{\,2}_1(ka) + Y'^{\,2}_1(ka)}}$$

(31)

Numerical Results

Now we present the numerical results of the dispersion relation for various depths to radius ratios. We consider the following ratios: h/a = 1, h/a = 2, h/a = 5. Table I describes the values of ka for which

$$\frac{\sigma^2 a}{g} - ka\tanh\left(ka\frac{h}{a}\right) = 0.$$

Figure I presents the dispersion relation as a function ka for h/a = 1 for various $\frac{\sigma^2 a}{g}$. Zeros can be observed from this graph which matches with the values mentioned in table I.

Now we present the computational results for the first order non-dimensional horizontal force component given by equation (30). Horizontal component is non-dimensionalized by dividing it by $\rho g a^2 A$. Various depth to radius ratios we consider are h/a = 1, h/a = 2, h/a = 5. Figure II and figure III depict the non-dimensional horizontal force component for $\sigma t = \pi/6$ and $\sigma t = \pi/4$.

We display the graphical results for the non-dimensional horizontal force component (30) for various σt for a particular depth to radius ratio, h/a = 2. It has been displayed in figure IV.

Now we present the non-dimensional maximum horizontal force component given by the equation (31). For h/a = 1, h/a = 2, h/a = 5. Figure V depicts the maximum non-dimensional horizontal force component.

From all the above results it is obvious that the force component is more effective when the parameter ka ranges between 0.5 and 1.5. Larger the depth is larger the force. The effect is negligible for large ka.

Table I

Zeros of Dispersion relation (27) for various $\frac{\sigma^2 a}{g}$ and *h/a*

$\frac{\sigma^2 a}{g}$ → $\frac{h}{a}$ ↓	1	2	3	5
1	1.199679	2.065338	3.014483	5.000454
2	1.032669	2.001335	3.000037	5.000000
5	1.000091	2.000000	3.000000	5.000000

Figure 1.
Zeros of Dispersion relation (27) for various σt with h/a = 1

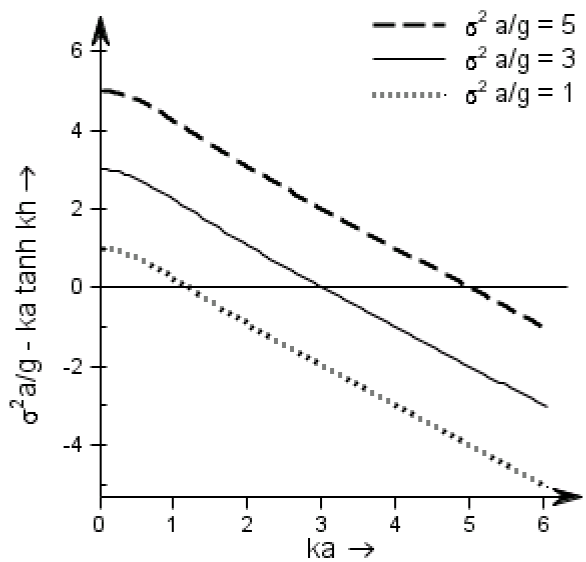


Figure II
Non-dimensional horizontal force component for various h/a with $\sigma t = \pi / 6$

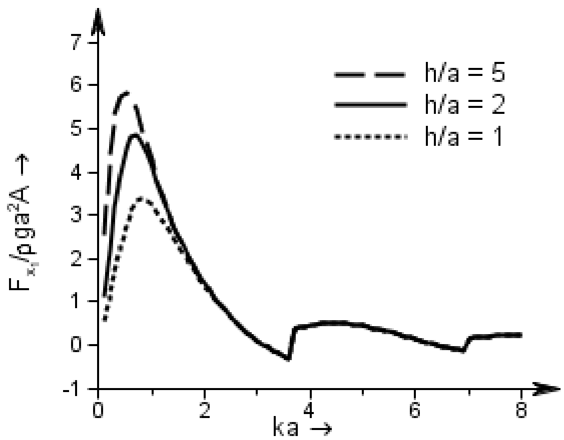


Figure III
Non-dimensional horizontal force component for various h/a with $\sigma t = \pi / 4$

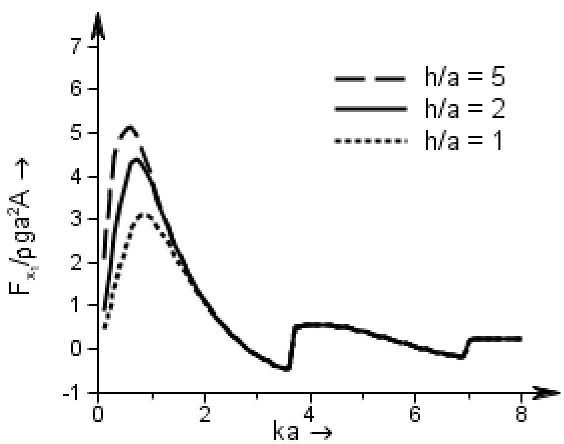


Figure IV
Non-dimensional horizontal force component for various σt
with h/a = 2.0

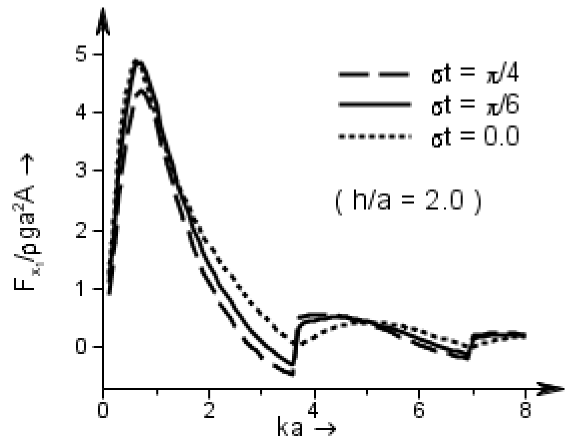
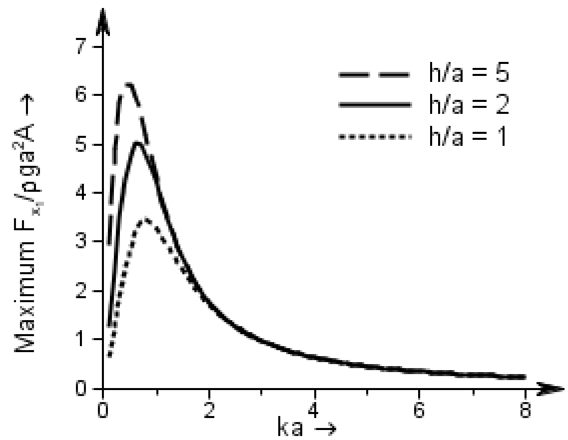


Figure V
Non-dimensional maximum horizontal force component for various h/a.



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