A New Constructive Proof of Graham's Theorem and More New Classes of Functionally Complete Functions

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Abstract

An n-valued two-variable truth function is called functionally complete, if all n-valued functions of m variables can be expressed in terms of the finite compositions of this function. R. L. Graham, using an indirect proof, proved the existence of n-valued ($n > 3$) functionally complete truth functions of two variables. He found one class of functionally complete functions. In this article, we will provide a constructive direct proof of Graham's Theorem. Using the same procedure, we found several more classes of functionally complete functions. The method and strategy of our proof can be widely used to find more new types of functionally complete functions.

1 Introduction

It is well known that there exist two and only two functionally complete Sheffer stroke functions, $(\neg P) \wedge (\neg Q)$,and $(\neg P) \vee (\neg Q)$, of 2-valued prepositional calculus. (i.e. all 2^{2^m} two-valued truth functions of m variables can be defined in terms of finite compositions of any one of these two Sheffer stroke functions. For detailed discussion of this concept, see [2]) R. L. Graham extended the result to n-valued truth functions. He proved the existence of functionally complete functions for $n \geq 2$. First, he proved all n-valued functions of m variables can be defined in terms of finite compositions of n-valued functions of two variables. Then, he defined a class of n-valued functions of two variables, and using an indirect proof, proved all n-valued functions of two variables can be defined in terms of finite compositions of any one of these functions in the class. This implies that all n-valued functions of m variables can be defined in terms of finite compositions of any one of these functions in the class. Therefore, all functions in this class are functionally complete (cf. [1]).If $T(n)$ denotes the number of n-valued functionally complete truth functions of two variables, it is known that $T(2) = 2$, $T(3) = 3774$ (cf. [3, 4]). For $n > 3$, it is still unknown. Graham showed there are $n!n^{n^2-2n}$ functions in his class. For example, when $n = 3$, there are 162 such functions. Comparing with $T(3) = 3774$, there should be many other types of functionally complete functions. In order to find more functionally complete functions, we need to have a method to examine the completeness directly. In the next section, we will give a direct constructive proof of the completeness of the Graham's theory. The direct proof is much easier to understand and can be widely used on proofs of other types of functionally complete functions.

2 A New Constructive Proof of Graham's Theorem

In order to compare our proof with Graham's proof, we will use most of the same notation as in Graham's article (cf.[1]). Here are some notations:

$$I = \{ 0, 1, \ldots\ldots, n-1 \}, \text{ where } n \geq 2.$$

$$\mathbb{C} = \{ \text{all n-valued truth functions of two variables from } I \times I \text{ into } I \}.$$

$\mathbb{C}$ has cardinality n^{n^2}.

If $F \in \mathbb{C}$, P(F) denote the set of all truth functions of two variables which can be defined in terms of finite compositions of the 2-variable function F(p,q).

The notation $[a_1, a_2, \ldots, a_{n^2}] \in P(F)$ implies that the function G (p,q) with the truth table

$$[G(0, 0); G(0, 1), \ldots, G(0, n-1); G(1, 0) \ldots\ldots G(n-1; n-1)] = [a_1, a_2, \ldots, a_{n^2}]$$

belongs to P(F).

Let ω be the bijection from $\{ 0, 1, \ldots\ldots, n^2 \}$ into $I \times I$ such that $\omega(1) = (0; 0)$; $\omega(2) = (0; 1), \ldots\ldots$, $\omega(n^2) = (n-1, n-1)$. So, If $\omega(k) = (p, q)$, then $a_k = G(\omega(k)) = G(p,q)$.

A function F is called functionally complete, if $P(F) = \mathbb{C}$. In other words, *F* is functional complete if any truth table $[a_1, a_2, \ldots, a_{n^2}]$ in $\mathbb{C}$ belongs to P(F).

In order to define a functionally complete function, we introduce two mappings. Let σ and π be arbitrary fixed mappings of I into I such that for all $a \in I$:

(i) $0 < r < n$ implies $\sigma^{(r)}(a) \neq a$ (where $\sigma^{(r)}(a)$ denotes the r^{th} iterate $\underbrace{\sigma(\sigma(\ldots\sigma}_{r}(a)\ldots))$, $\sigma^{(0)}(a) = a$. Note that σ is just a permutation of I, that consists of a single cycle.

(ii) There exists $r > 0$ such that $\pi^{(r)}(a) = 0$.

Now, we define the function from $I \times I$ into I:

$$F_{\sigma,\pi}(p,q) = \begin{cases} \sigma(p), & \text{if } p = q; \\ 0, & \text{if } p = 0 \text{ and } q \neq 0; \\ \pi(p), & \text{if } p \neq 0 \text{ and } q = 0; \\ \text{any value} & \text{otherwise;} \end{cases}$$

Abbreviate $F_{\sigma,\pi}(p,q)$ by F. Let $T_r = \{ t_1, t_2, \ldots, t_r \}$ be a subset of $\{ 1, 2, \ldots, n^2 \}$. For $b_i \in I$, the notation $B_r = [b_1, b_2, \ldots, b_r]_{T_r} \in P(F)$ will indicate there exists $[a_1, a_2, \ldots, a_{n^2}] \in P(F)$ such that $a_{t_i} = b_i$ for $1 \leq i \leq r$.

Graham's Theorem: The above function $F_{\sigma,\pi}$ (p,q) is functionally complete. (i.e. $P(F_{\sigma,\pi}) = \mathbb{C}$ or all truth tables $[b_1, b_2, ..., b_{n^2}]$ in $\mathbb{C}$ belong to P(F).)

Before proceeding to the proof of the theorem, we prove some lemmas.

Lemma 1: Assunme $T_k = [t_1, t_2, \ldots, t_k] \subset \{1, 2, \ldots, n^2\}$ is nonempty,. If $[b_1, b_2, \ldots, b_k]_{T_k} \in P(F)$, then $[\sigma^{(m)}(b_1), \sigma^{(m)}(b_2), \ldots, \sigma^{(m)}(b_k)]_{T_k} \in P(F)$ for all m.

Proof: When m = 0, $[\sigma^{(0)}(b_1), \sigma^{(0)}(b_2), \ldots, \sigma^{(0)}(b_k)]_{T_k} = [b_1, b_2, \ldots, b_k]_{T_k} \in P(F)$.

If $[\sigma^{(m)}(b_1), \sigma^{(m)}(b_2), \ldots, \sigma^{(m)}(b_k)]_{T_k} \in P(F)$,

then $F([\sigma^{(m)}(b_1), \sigma^{(m)}(b_2), \ldots, \sigma^{(m)}(b_k)]_{T_k}, [\sigma^{(m)}(b_1), \sigma^{(m)}(b_2), \ldots, \sigma^{(m)}(b_k)]_{T_k}) = [\sigma^{(m+1)}(b_1), \sigma^{(m+1)}(b_2), \ldots, \sigma^{(m+1)}(b_k)]_{T_k} \in P(F)$.

By induction, the lemma is true. □

Let $\mathbf{A_k}$ $(1 \le k \le n^2)$ be the statement: For any $T_k = [t_1, t_2, \ldots, t_k] \subset \{1, 2, \ldots, n^2\}$, all truth tables $B = [b_1, b_2, \ldots, b_k]_{T_k} \in P(F)$, for arbitrary $b_i \in I$, $1 \le i \le k$.

If we prove that $\mathbf{A}_{n^2}$ is true, i.e. all truth tables $B = [b_1, b_2, \ldots, b_{n^2}] \in P(F)$, for any $b_i \in I$, $1 \le i \le n^2$, then F is functionally complete. The proof of the Graham's theorem will be complete.

Here is the idea of our proof: First, prove $\mathbf{A_1}$ is true, then prove $\mathbf{A_2}$ is true, and finally, by induction, prove $\mathbf{A_3}, \ldots, \mathbf{A}_{n^2}$ are true.

First, prove the statement $\mathbf{A_1}$:

For any $t_1 \in I$, let $T_1 = \{t_1\}$. There exists a $[a]_{T_1} \in P(F)$, for example, a = F (p, q), where $\omega(t_1) = (p,q)$.

By Lemma 1, $[\sigma(a)]_{T_1} \in P(F)$, $[\sigma^{(2)}(a)]_{T_1} \in P(F)$, …, $[\sigma^{(n-1)}(a)]_{T_1} \in P(F)$. Since the function σ is a single cycle permutation, we proved for any $b_i \in I$, $[b_i]_{T_1} \in P(F)$. $\mathbf{A_1}$ has been proved. □

Before we prove the statement A_2, we need to prove two lemmas.

Lemma 2: For any $T_2 = \{t_1, t_2\} \subset \{1, 2, \ldots, n^2\}$, $t_1 \neq t_2$, there exist two numbers a and b in I, such that $a \neq b$ and $[a, b]_{T_2} \in P(F)$.

Proof: For any $p \in I$, and $q \in I$, consider these two functions: $K(p, q) = F(p, p) = \sigma(p) \in P(F)$, $H(p, q) = F(q, q) = \sigma(q)$. These two functions are defined in terms of F. Therefore, they belong to P(F).

Assume $\omega(t_1) = (m_1, n_1)$ and $\omega(t_2) = (m_2, n_2)$. Since $t_1 \neq t_2$, we have either $m_1 \neq m_2$ or $n_1 \neq n_2$.

If $m_1 \neq m_2$, then the truth table of K(p, q) = F(p, p) on T_2 will be $[\sigma(m_1), \sigma(m_2)]_{T_1} \in P(F)$. Since $m_1 \neq m_2$, we have $\sigma(m_1) \neq \sigma(m_2)$.

If $n_1 \neq n_2$, then the truth table of H(p, q) = F(q, q) on T_2 will be $[\sigma(n_1), \sigma(n_2)]_{T_1} \in P(F)$. Since $n_1 \neq n_2$, we have $\sigma(n_1) \neq \sigma(n_2)$.

Lemma 2 is proved. □

Lemma 3: For any $T_2 = \{ t_1, t_2 \} \subset \{ 1, 2, \ldots, n^2 \}$, $t_1 \neq t_2$, There exists $c \neq 0$ in I, and $d \neq 0$ in I such that

(i) $[c, 0]_{T_2} \in P(F)$.

(ii) $[0, d]_{T_2} \in P(F)$.

(iii) $[0, 0]_{T_2} \in P(F)$.

Proof: Assume $[a, b]_{T_2} \in P(F)$ and $a \neq b$. There exist s and t such that $\sigma^{(s)}(a) = 0$, and $\sigma^{(t)}(b) = 0$. Since $a \neq b$, $\sigma^{(s)}(b) \neq 0$, $\sigma^{(t)}(a) \neq 0$. By *Lemma* 1, $[\sigma^{(s)}(a), \sigma^{(s)}(b)]_{T_2} = [0, \sigma^{(s)}(b)]_{T_2} \in P(F)$, and $[\sigma^{(t)}(a), \sigma^{(t)}(b)]_{T_2} = [\sigma^{(t)}(a), 0]_{T_2} \in P(F)$. Let $c = \sigma^{(t)}(a)$, and $d = \sigma^{(s)}(b)$. (i) and (ii) are proved.

Since there exist $c \neq 0$and $d \neq 0$. $[c, 0]_{T_2} \in P(F)$ and $[0 , d]_{T_2} \in P(F)$.

$$F([0, d]_{T_2}, [c , 0]_{T_2}) = [0, \pi(d)]_{T_2} \in P(F).$$

If $\pi(d) = 0$, then we have $[0, 0]_{T_2} \in P(F)$. If $\pi(d) \neq 0$,

$$F([0, \pi(d)]_{T_2}, [c , 0]_{T_2}) = [0, \pi^{(2)}(d)]_{T_2} \in P(F).$$

If $\pi^{(2)}(d) = 0$, then $[0, 0]_{T_2} \in P(F)$, otherwise, we will continue the above procedure to get

$$F([0, \pi^{(2)}(d)]_{T_2}, [c , 0]_{T_2}) = [0, \pi^{(3)}(d)]_{T_2} \in P(F),$$

and so on. Eventually, according to the property of π, there is a number r such that $\pi^{(r)}(d) = 0$, and $[0, \pi^{(r)}(d)]_{T_2} = [0, 0]_{T_2} \in P(F)$. (iii) is proved. Therefore *Lemma* 3 is true. □

Now, we prove $\mathbf{A_2}$ is true:

For any $T_2 = \{ t_1, t_2 \} \subset \{ 1, 2, \ldots, n^2 \}$, by *Lemma* 3 (iii), $[0, 0]_{T_2} \in P(F)$. By *Lemma* 1, we know that $[\sigma(0), \sigma(0)]_{T_2} \in P(F)$.

$[\sigma^{(2)}(0), \sigma^{(2)}(0)]_{T_2} \in P(F), \ldots, [\sigma^{(n-1)}(0), \sigma^{(n-1)}(0)]_{T_2} \in P(F)$.

By *Lemma* 3 (i), there exists $c \in I$, such that $c \neq 0$ and $[\, c, 0\,]_{T_2} \in P(F)$. By *Lemma* 3 (iii), $[\, 0, 0\,]_{T_2} \in P(F)$. Therefore,

$$F(\,[\,0, 0\,]_{T_2}, [\,c, 0\,]_{T_2}\,) = [\,0, \sigma(0)\,]_{T_2} \in P(F);$$

$$F([\,0, \sigma(0)\,]_{T_2}, [\,\sigma(0), \sigma(0)\,]_{T_2}\,) = [\,0, \sigma^{(2)}(0)\,]_{T_2} \in P(F);$$

$$F([\,0, \sigma^{(2)}(0)\,]_{T_2},\ [\,\sigma^{(2)}(0), \sigma^{(2)}(0)\,]_{T_2}\,) = [\,0, \sigma^{(3)}(0)\,]_{T_2} \in P(F);$$

$$\ldots\ldots;$$

$$F([\,0, \sigma^{(n-2)}(0)\,]_{T_2},\ [\,\sigma^{(n-2)}(0), \sigma^{(n-2)}(0)\,]_{T_2}\,) = [\,0, \sigma^{(n-1)}(0)\,]_{T_2} \in P(F).$$

According to the property of σ, we know that for all $b \in I$, $[\,0, b\,]_{T_2} \in P(F)$. By *Lemma* 1, for any $s \in I$, and any $b \in I$,
$[\,\sigma^{(s)}(0), \sigma^{(s)}(b)\,]_{T_2} \in P(F)$. That is: for any $s \in I$, for any $b \in I$, $[\,\sigma^{(s)}(0), b\,]_{T_2} \in$ P(F). This implies for any $a \in I$ and $b \in I$,
$[\,a, b\,]_{T_2} \in P(F)$. $\mathbf{A_2}$ has been proved. □

Now, we prove $\mathbf{A_k}$ by induction, for all k where $3 \leq k \leq n^2$:
Assume $\mathbf{A_m}$ is true for all $m \leq k$, where $k \geq 2$, we are going to prove that $\mathbf{A_{k+1}}$ is true. We abbreviate truth table $[\,b_1, b_2, \ldots, b_k\,]_{T_k}$ by B_K. and denote $[\,\sigma^{(s)}(b_1), \sigma^{(s)}(b_2), \ldots, \sigma^{(s)}(b_k)\,]_{T_k}$ By $\sigma^{(s)}(B_k)$. We will prove that for any B_k on any set $T_k = \{\, t_1, t_2, \ldots, t_k \,\} \subset T_{k+1} \subset \{\, 1, 2, \ldots, n^2 \,\}$, and any $\beta \in I$, the truth table $[\,B_k, \beta\,]_{T_{k+1}} \in P(F)$.

We need to prove some lemmas:

Lemma 4: For any sets $T_{k+1} = \{\, t_1, t_2, \ldots, t_{k+1} \,\} \subset \{\, 1, 2, \ldots, n^2 \,\}$, and any $B_{k-1} = [\,b_1, b_2, \ldots, b_{k-1}\,]_{T_{k-1}}$, where $T_{k-1} = \{\, t_1, t_2, \ldots, t_{k-1} \,\}$, there exists an integer s, such that $[\,\sigma^{(s)}(B_{k-1}), 0, 0\,]_{T_{k+1}} \in P(F)$.

Proof: Since $\mathbf{A_k}$ is true, there exist $\alpha \in I$ and $\beta_0 \in I$, such that $[\,B_{k-1}, \alpha, 0\,]_{T_{k+1}} \in P(F)$ and $[\,B_{k-1}, 0, \beta_0\,]_{T_{k+1}} \in P(F)$. If $\alpha = 0$ or $\beta_0 = 0$, then *Lemma* 4 is true, with $s = 0$. If not, then $\alpha \neq 0$ and $\beta_0 \neq 0$. Consider

$$F(\,[\,B_{k-1}, \alpha, 0\,]_{T_{k+1}}, [\,B_{k-1}, 0, \beta_0\,]_{T_{k+1}}\,) = [\,\sigma(\,B_{k-1}\,), \pi(\,\alpha\,), 0\,]_{T_{k+1}} \in P(F).$$

If $\pi(\,\alpha\,) = 0$, then *Lemma* 4 is proved with $s = 1$. Otherwise there exists a number $\beta_1 \in I$, such that $[\,\sigma(\,B_{k-1}), 0, \beta_1\,]_{T_{k+1}} \in P(F)$. If $\beta_1 = 0$, then *Lemma* 4 is true, with $s = 1$. Otherwise, if $\beta_1 \neq 0$, then

$$F([\,\sigma(B_{k-1}), \pi(\alpha), 0\,]_{T_{k+1}},\ [\,\sigma(B_{k-1}), 0, \beta_1\,]_{T_{k+1}}) = [\,\sigma^{(2)}(B_{k-1}), \pi^{(2)}(\alpha), 0\,]_{T_{k+1}} \in P(F).$$

If $\pi^{(2)}(\alpha) = 0$, then *Lemma* 4 is proved with s =2. Otherwise continue the above procedure. Eventually, there exists an integer s such that either $\beta_s = 0$, or $\pi^{(s)}(\alpha) = 0$. Therefore, $[\,\sigma^{(s)}(B_{k-1}), 0, 0\,]_{T_{k+1}} \in P(F)$ for some natural number s. □

Lemma 5: For any set $T_{k+1} = \{ t_1, t_2, \ldots, t_{k+1} \} \subset \{ 1, 2, \ldots, n^2 \}$, and any $B_{k-1} = [\,b_1, b_2, \ldots, b_{k-1}\,]_{T_{k-1}}$, where $T_{k-1} = \{ t_1, t_2, \ldots, t_{k-1} \} \subset T_{k+1}$, if $[\,\sigma^{(s)}(B_{k-1}), 0, 0\,]_{T_{k+1}} \in P(F)$, then $[\,\sigma^{(s+1)}(B_{k-1}), 0, 0\,]_{T_{k+1}} \in P(F)$.

Proof: Let's discuss the following two cases:

Case 1: There exist $\alpha \neq 0$, and $\beta \neq 0$, such that $[\,\sigma^{(s)}(B_{k-1}), \alpha, \beta\,]_{T_{k+1}} \in P(F)$.

Case 2: If Case 1 is not true, then for any α and β, $[\,\sigma^{(s)}(B_{k-1}), \alpha, \beta\,]_{T_{k+1}} \in P(F)$ implies that either $\alpha = 0$ or $\beta = 0$.

Proof of Case 1: If there exist $\alpha \neq 0$, and $\beta \neq 0$ such that $[\,\sigma^{(s)}(B_{k-1}), \alpha, \beta\,]_{T_{k+1}} \in P(F)$, then

$$F([\,\sigma^{(s)}(B), 0, 0\,]_{T_{k+1}},\ [\,\sigma^{(s)}(B_{k-1}), \alpha, \beta\,]_{T_{k+1}}) = [\,\sigma^{(s+1)}(B_{k-1}), 0, 0\,]_{T_{k+1}} \in P(F).$$

Proof of Case 2: For any $\alpha \neq 0$, there exists a number $\beta \in I$, such that $[\,\sigma^{(s)}(B_{k-1}), \alpha, \beta\,]_{T_{k+1}} \in P(F)$. Since $\alpha \neq 0$, β must be 0. Therefore, for all $\alpha \neq 0$, $[\,\sigma^{(s)}(B_{k-1}), \alpha, 0\,]_{T_{k+1}} \in P(F)$. Similarly, we have that for all $\beta \neq 0$ $[\,\sigma^{(s)}(B_{k-1}), 0, \beta\,]_{T_{k+1}} \in P(F)$. By choosing some α and β, such that $\alpha \neq 0$, $\pi(\alpha) = 0$ and $\beta \neq 0$, we have that

$$F([\,\sigma^{(s)}(B_{k-1}), \alpha, 0\,]_{T_{k+1}},\ [\,\sigma^{(s)}(B_{k-1}), 0, \beta\,]_{T_{k+1}}) = [\,\sigma^{(s+1)}(B_{k-1}), 0, 0\,]_{T_{k+1}} \in P(F).$$ □

Lemma 6: For any $B_{k-1} = [\,b_1, b_2, \ldots, b_{k-1}\,]_{T_{k-1}}$, where $T_{k-1} = \{ t_1, t_2, \ldots, t_{k-1} \} \subset T_{k+1}$, if $[\,\sigma^{(s)}(B_{k-1}), 0, 0\,]_{T_{k+1}} \in P(F)$, for some $s \in I$, then $[\,B_{k-1}, \alpha, \alpha\,]_{T_{k+1}} \in P(F)$, for all $\alpha \in I$.

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Proof: Since $[\,\sigma^{(s)}(B_{k-1}), 0, 0\,]_{T_{k+1}} \in P(F)$, by *Lemma* 5, we have $[\,\sigma^{(s+1)}(B_{k-1}), 0, 0\,]_{T_{k+1}} \in P(F)$. Continuously using *Lemma* 5, we have that $[\,\sigma^{(s+2)}(B_{k-1}), 0, 0$

$]_{T_{k+1}} \in P(F)$. $[\ \sigma^{(s+3)}(B_{k-1}), 0, 0\ ]_{T_{k+1}} \in P(F)$, ……, $[\ \sigma^{(s+n-1)}(B_{k-1}), 0, 0\ ]_{T_{k+1}} \in P(F)$. This means $[\ \sigma^{(r)}(B_{k-1}), 0, 0\ ]_{T_{k+1}} \in P(F)$ for all r.

For any $\alpha = \sigma^{(t)}(\ 0\ )$, for some t, by the result above, let r = n-t, $[\sigma^{(n-t)}(B_{k-1}), 0, 0\ ]_{T_{k+1}} \in P(F)$. By *Lemma* 1, $[\ \sigma^{(n)}(B), \sigma^{(t)}(\ 0\ ), \sigma^{(t)}(0)\ ]_{T_{k+1}} \in P(F)$. Therefore, $[\ B_{k-1}, \alpha, \alpha\ ]_{T_{k+1}} \in P(F)$. □

By *Lemma* 4, and *Lemma* 6, we will get *Lemma* 7 immediately:

Lemma 7: For any $B_{k-1} = [\ b_1, b_2, \ldots, b_{k-1}\ ]_{T_{k-1}}$, and any $\alpha \in I$, we have $[\ B_{k-1}, \alpha, \alpha\ ]_{T_{k+1}} \in P(F)$.

This implies that if $b_i = b_j$ for some $i \neq j$, then $B_{k+1} = [\ b_1, b_2, \ldots b_i, \ldots b_j, \ldots, b_{k+1}\ ]_{T_{k+1}} \in P(F)$.

Now, we assume that A_k is true, where $k \geq 2$, and we going to prove for any $B_k = [\ b_1, b_2, \ldots, b_k\ ]_{T_k}$, and any $\beta \in I$, $[\ B_k, \beta\ ]_{T_{k+1}} \in P(F)$. This implies that $\mathbf{A_{k+1}}$ is true.

Proof: For any $B_k = [\ b_1, b_2, \ldots, b_k\ ]_{T_k}$, and any $\beta = \sigma^{(t)}(\ 0\ ) \in I$ for some t, If $b_i = b_j$, for some $i \neq j$, then by *Lemma* 7, $[\ B_k, \beta\ ]_{T_{k+1}} \in P(F)$. If all b_i's are distinct, there exists a number γ, such that $[\ B_k, \gamma\ ]_{T_{k+1}} \in P(F)$. Assume that $\sigma^{(s)}(\ \gamma\ ) = 0$, for some s. Since all b_i's are distinct, there exists some $b_i \neq \gamma$. Without loosing of generality, assume that $b_k \neq \gamma$. We have $B_k = [\ B_k, b_k\ ]_{T_k}$, and $[\ B_{k-1}, b_k, \gamma\ ]_{T_{k+1}} \in P)$. Since $\sigma^{(s)}(\ \gamma\ ) = 0$, by *Lemma* 1, we have that $[\sigma^{(s)}(\ B_{k-1}\ ), \sigma^{(s)}(\ b_k\ ), \sigma^{(s)}(\ \gamma\ )\ ]_{T_{k+1}} = [\sigma^{(s)}(B_{k-1}), \sigma^{(s)}(\ b_k\ ), 0\ ]_{T_{k+1}} \in P(F)$. By *Lemma* 7, we have $[\sigma^{(s)}(\ B_{k-1}\ ), \sigma^{(s)}(\ b_k\ ), \sigma^{(s)}(\ b_k\ )\ ]_{T_{k+1}} \in P(F)$. Since $b_k \neq \gamma$, and $\sigma^{(s)}(\ \gamma\ ) = 0$, so, $\sigma^{(s)}(\ b_k\ ) \neq 0$.

$F([\sigma^{(s)}(\ B_{k-1}\ ), \sigma^{(s)}(\ b_k\ ), 0\ ]_{T_{k+1}}, [\sigma^{(s)}(\ B_{k-1}\ ), \sigma^{(s)}(\ b_k\ ), \sigma^{(s)}(\ 0\ )\ ]_{T_{k+1}}) = [\sigma^{(s+1)}(\ B_{k-1}\ ), \sigma^{(s+1)}(\ b_k\ ), 0\ ]_{T_{k+1}} \in P(F)$.

$F([\sigma^{(s+1)}(\ B_{k-1}\ ), \sigma^{(s+1)}(\ b_k\ ), 0\ ]_{T_{k+1}}, [\sigma^{(s+1)}(\ B_{k-1}\ ), \sigma^{(s+1)}(\ b_k\ ), \sigma(\ 0\ )\ ]_{T_{k+1}})$
$= [\sigma^{(s+2)}(\ B_{k-1}\ ), \sigma^{(s+2)}(\ b_k\ ), 0\ ]_{T_{k+1}} \in P(F)$.

$F([\sigma^{(s+2)}(\ B_{k-1}\ ), \sigma^{(s+2)}(\ b_k\ ), 0\ ]_{T_{k+1}}, [\sigma^{(s+2)}(\ B_{k-1}\ ), \sigma^{(s+2)}(\ b_k), \sigma(\ 0\ )\ ]_{T_{k+1}})$
$= [\sigma^{(s+3)}(\ B_{k-1}\ ), \sigma^{(s+3)}(\ \alpha\ ), 0\ ]_{T_{k+1}} \in P(F)$.

……,

Continue this, we will have $[\sigma^{(s+j)}(B_{k-1}), \sigma^{(s+j)}(\alpha), 0]_{T_{k+1}} \in P(F)$, for any number j. Since $\beta = \sigma^{(t)}(0)$, choose j such that $s + j = n – t$. We have $[\sigma^{(n-t)}(B_{k-1}), \sigma^{(n-t)}(b_k), 0]_{T_{k+1}} \in P(F)$, *By Lemma* 1, we have

$[\sigma^{(n)}(B_{k-1}), \sigma^{(n)}(b_k), \sigma^{(t)}(0)]_{T_{k+1}} = [B_{k-1}, b_k, \beta]_{T_{k+1}} = [B_k, \beta]_{T_{k+1}} \in P(F)$

$\mathbf{A_{k+1}}$ is proved. □

Therefore, $\mathbf{A_k}$ is true, for all k. Especially, $\mathbf{A}_{n^2}$ is true. This implies that all truth tables $[b_1, b_2,..., b_{n^2}]$ in $\mathbb{C}$ belong to P(F). The Graham's theorem is proved.

3 More New Classes of Functionally Complete Functions

The following functions are functionally complete. The proofs of completeness of these functions are similar to the above proof of the Graham's Theorem, therefore, we only give the proof of the first function.

1.

$$F_1(p,q) = \begin{cases} \sigma^{(i+1)}(0), & \text{if } \sigma^{(i)}(0) = p = q \neq 0; \\ \sigma^{(\max\{i,j\})}(0), & \text{if } p = \sigma^{(i)}(0) \text{ and } q = \sigma^{(j)}(0), i \neq j, 0 \le i, j \le n-1; \end{cases}$$

2. If we change the max { i, j } of the function F_1 to min { i, j }, we will have another functionally complete function:

$$F_2(p,q) = \begin{cases} \sigma^{(i+1)}(0), & \text{if } \sigma^{(i)}(0) = p = q \neq 0; \\ \sigma^{(\min\{i,j\})}(0), & \text{if } p = \sigma^{(i)}(0) \text{ and } q = \sigma^{(j)}(0), i \neq j, 0 \le i, j \le n-1; \end{cases}$$

3. If we change the max { i, j } of the function F_1 to i+j, we will have another functionally complete function:

$$F_3(p,q) = \begin{cases} \sigma^{(i+1)}(0), & \text{if } \sigma^{(i)}(0) = p = q \neq 0; \\ \sigma^{(i+j)}(0), & \text{if } p = \sigma^{(i)}(0) \text{ and } q = \sigma^{(j)}(0), i \neq j, 0 \le i, j \le n-1; \end{cases}$$

4. If n is prime, and $n \ge 3$, the following function is functionally complete.

$$F_4(p,q) = \begin{cases} \sigma(p), & \text{if } p = q; \\ \sigma(q), & \text{if } p = 0 \text{ and } q \neq 0; \\ \sigma(p), & \text{if } p \neq 0 \text{ and } q = 0; \\ \text{any value}, & \text{otherwise}; \end{cases}$$

5.

$$F_5(p,q) = \begin{cases} \sigma^{(i+1)}(0), & \text{if } \sigma^{(i)}(0) = p = q \neq 0; \\ \pi(\sigma^{(i)}(0)), & \text{if } p = \sigma^{(i)}(0) \neq 0 \text{ and } q = 0; \\ \sigma^{(i+1)}(0), & \text{if } p = 0 \text{ and } q = \sigma^{(i)}(0) \neq 0; \\ \sigma(0), & \text{if } p = \sigma(0) \text{ and } p \neq q \text{ and } q = \sigma^{(i)}(0) \neq 0; \\ \sigma(0), & \text{if } q = \sigma(0) \text{ and } p \neq q \text{ and } p = \sigma^{(i)}(0) \neq 0; \\ \text{any value}, & \text{otherwise}; \end{cases}$$

6.

$$F_6(p,q) = \begin{cases} \sigma^{(i+1)}(0), & \text{if } \sigma^{(i)}(0) = p = q; \\ \pi(\sigma^{(i)}(0)), & \text{if } p = \sigma^{(i)}(0) \neq 0 \text{ and } q = 0; \\ \sigma^{(i+1)}(0), & \text{if } p = 0 \text{ and } q = \sigma^{(i)}(0) \neq 0; \\ \sigma^{(\max\{i,j\})}(0), & \text{if } p = \sigma^{(i)}(0) \neq 0, \text{ and } q = \sigma^{(j)}(0) \neq 0, \text{ and } p \neq q; \end{cases}$$

As an example, we only prove the first function:

Theorem. The function F_1 is functionally complete, where

$$F_1(p,q) = \begin{cases} \sigma^{(i+1)}(0), & \text{if } \sigma^{(i)}(0) = p = q \neq 0; \\ \sigma^{(\max\{i,j\})}(0), & \text{if } p = \sigma^{(i)}(0) \text{ and } q = \sigma^{(j)}(0), i \neq j, 0 \leq i, j \leq n-1; \end{cases}$$

We will follow the same procedure as the proof of the *Graham's Theorem*: *Lemma* 1, A_1, *Lemma* 2, *Lemma* 3, A_2, *Lemma* 4, *Lemma* 5, *Lemma* 6, *Lemma* 7 and A_{k+1}. All lemmas state exactly the same as lemmas in the last section. If the proof is also the same, we will not repeat. Abbreviate the function $F_1(p,q)$ by F.

Lemma 1: Assunme $T_k = [t_1, t_2, \ldots, t_k] \subset \{1, 2, \ldots, n^2\}$ is nonempty,. If $[b_1, b_2, \ldots, b_k]_{T_k} \in P(F)$, then $[\sigma^{(m)}(b_1), \sigma^{(m)}(b_2), \ldots, \sigma^{(m)}(b_k)]_{T_k} \in P(F)$ for all m.

Proof: The same as the proof of *Lemma* 1 in the last section.

Let $\mathbf{A_k}$ ($1\le k \le n^2$) be the statement: For any $T_k = [t_1, t_2, \ldots, t_k] \subset \{ 1, 2, \ldots, n^2 \}$, all truth tables $B = [b_1, b_2, \ldots, b_k]_{T_k} \in P(F)$, for arbitrary $b_i \in I$, $1 \le i \le k$.

Proof of $\mathbf{A_1}$: The same as the proof of $\mathbf{A_1}$ in the last section.

Lemma 2: For any $T_2 = \{ t_1, t_2 \} \subset \{ 1, 2, \ldots, n^2 \}$, $t_1 \ne t_2$, there exist two numbers a and b in I, such that $a \ne b$ and $[a, b]_{T_2} \in P(F)$.

Proof: The same as the proof of *Lemma* 2 in the last section.

Lemma 3: For any $T_2 = \{ t_1, t_2 \} \subset \{ 1, 2, \ldots, n^2 \}$, $t_1 \ne t_2$, There exists $c \ne 0$ in I, and $d \ne 0$ in I such that

(i) $[c, 0]_{T_2} \in P(F)$.

(ii) $[0, d]_{T_2} \in P(F)$.

(iii) $[0, 0]_{T_2} \in P(F)$.

Proof: The proof of case (i) and (iii) are the same as the proof of case (i) and (ii) of *Lemma* 3 in the last section. We now prove case (iii):
By case (i) and (ii) of *Lemma* 3 and *Lemma* 1, there exist $s \ne n\text{-}1$ and $t \ne n\text{-}1$ such that

$$[\sigma^{(n-1)}(0), \sigma^{(s)}(0)]_{T_2} \in P(F), \text{ and } [\sigma^{(t)}(0), \sigma^{(n-1)}(0)]_{T_2} \in P(F);$$

$$F([\sigma^{(n-1)}(0), \sigma^{(s)}(0)]_{T_2}, [\sigma^{(t)}(0), \sigma^{(n-1)}(0)]_{T_2}) = [\sigma^{(n-1)}(0), \sigma^{(n-1)}(0)]_{T_2} \in P(F);$$

By *Lemma* 1, $[0, 0]_{T_2} \in P(F)$. □

Proof of $\mathbf{A_2}$:

Proof: By Lemma 3 (i), there exists s such that $[\sigma^{(s)}(0), 0]_{T_2} \in P(F)$. By *Lemma* 3 (iii) and *Lemma* 1, we have $[\sigma^{(r)}(0), \sigma^{(r)}(0)]_{T_2} \in P(F)$, for all $0 \le r \le n\text{-}1$,.

$$F([\sigma^{(s)}(0), 0]_{T_2}, [\sigma(0), \sigma(0)]_{T_2}) = [\sigma^{(s)}(0),, \sigma(0)]_{T_2} \in P(F);$$

$$F([\sigma^{(s)}(0),, \sigma(0)]_{T_2}, [\sigma^{(2)}(0), \sigma^{(2)}(0)]_{T_2}) = [\sigma^{(s)}(0), \sigma^{(2)}(0)]_{T_2} \in P(F);$$

$$\ldots\ldots;$$

$$F([\sigma^{(s)}(0), 0]_{T_2}, [\sigma^{(s-1)}(0), \sigma^{(s-1)}(0)]_{T_2}) = [\sigma^{(s)}(0), \sigma^{(s-1)}(0)]_{T_2} \in P(F).$$

Since $[\sigma^{(s)}(0), 0]_{T_2} \in P(F)$, by *Lemma* 1, $[0, \sigma^{(n-s)}(0)]_{T_2} \in P(F)$.

$$F([0, \sigma^{(n-s)}(0)]_{T_2}, [\sigma(0), \sigma(0)]_{T_2}) = [\sigma(0), \sigma^{(n-s)}(0)]_{T_2} \in P(F).$$

By *Lemma* 1, $[\sigma^{(s)}(0), \sigma^{(n-1)}(0)]_{T_2} \in P(F)$.

$$F([0, \sigma^{(n-s)}(0)]_{T_2}, [\sigma^{(2)}(0), \sigma^{(2)}(0)]_{T_2}) = [\sigma^{(2)}(0), \sigma^{(n-s)}(0)]_{T_2} \in P(F).$$

By *Lemma* 1, $[\sigma^{(s)}(0), \sigma^{(n-2)}(0)\ ]_{T_2} \in P(F)$.

…… ;

$$F(\ [0, \sigma^{(n-s)}(0)\ ]_{T_2},\ [\ \sigma^{(n-s-1)}(0), \sigma^{(n-s-1)}(0)\ ]_{T_2}\ ) = [\sigma^{(n-s-1)}(0), \sigma^{(n-s)}(0)\ ]_{T_2} \in P(F).$$

By *Lemma* 1, $[\sigma^{(s)}(0), \sigma^{(s+1)}(0)\ ]_{T_2} \in P(F)$.

Also, we have $[\sigma^{(s)}(0), \sigma^{(s)}(0)\ ]_{T_2} \in P(F)$.

Therefore, for any $b \in I$, we have $[\sigma^{(s)}(0), b\ ]_{T_2} \in P(F)$. By Lemma 1, for any r, $[\sigma^{(r)}(\ \sigma^{(s)}(0)), \sigma^{(r)}(b)\ ]_{T_2} \in P(F)$. That is, for any a, b in I, $[a, b\ ]_{T_2} \in P(F)$. $\mathbf{A_2}$ is proved. □

Now, we prove $\mathbf{A_k}$ for all k by induction, where $3 \le k \le n^2$: Assume $\mathbf{A_m}$ is true for all $m \le k$, we are going to prove that $\mathbf{A_{k+1}}$ is true.

Lemma 4: For any sets $T_{k+1} = \{\ t_1, t_2, \ldots, t_{k+1}\ \} \subset \{\ 1, 2, \ldots, n^2\ \}$, and any $B_{k-1} = [\ b_1, b_2, \ldots, b_{k-1}\ ]_{T_{k-1}}$, where $T_{k-1} = \{\ t_1, t_2, \ldots, t_{k-1}\ \}$, there exists an integer s, such that $[\ \sigma^{(s)}(B_{k-1}), 0, 0\ ]_{T_{k+1}} \in P(F)$.

Proof: Since $\mathbf{A_k}$ is true, there exist $\alpha \in I$ and $\beta \in I$, such that $[\ B_{k-1}, \alpha, \sigma^{(n-1)}(0)\ ]_{T_{k+1}} \in P(F)$ and $[\ B_{k-1}, \sigma^{(n-1)}(0), \beta\ ]_{T_{k+1}} \in P(F)$. If $\alpha = \sigma^{(n-1)}(0)$ or $\beta = \sigma^{(n-1)}(0)$, then by *Lemma* 1, $[\ \sigma(\ B_{k-1}\ ),\ 0\ ,\ 0\ ]_{T_{k+1}} \in P(F)$. *Lemma* 4 is true, and s = 1. If not, then $\alpha \ne \sigma^{(n-1)}(0)$ and $\beta \ne \sigma^{(n-1)}(0)$. Consider

$$F(\ [\ B_{k-1}, \alpha, \sigma^{(n-1)}(0)\ ]_{T_{k+1}}, [\ B_{k-1}, \sigma^{(n-1)}(0), \beta\ ]_{T_{k+1}}\ ) = [\ \sigma(\ B_{k-1}\ ), \sigma^{(n-1)}(0), \sigma^{(n-1)}(0)\ ]_{T_{k+1}} \in P(F).$$

By *Lemma* 1, $[\ \sigma^{(2)}(\ B_{k-1}\ ),\ 0\ ,\ 0\ ]_{T_{k+1}} \in P(F)$. *Lemma* 4 is true, and s = 2. □

Lemma 5: For any set $T_{k+1} = \{\ t_1, t_2, \ldots, t_{k+1}\ \} \subset \{\ 1, 2, \ldots, n^2\ \}$, and any $B_{k-1} = [\ b_1, b_2, \ldots, b_{k-1}\ ]_{T_{k-1}}$, where $T_{k-1} = \{\ t_1, t_2, \ldots, t_{k-1}\ \} \subset T_{k+1}$, if $[\ \sigma^{(s)}(B_{k-1}), 0, 0\ ]_{T_{k+1}} \in P(F)$, then $[\ \sigma^{(s+1)}(B_{k-1}), 0, 0\ ]_{T_{k+1}} \in P(F)$.

Proof: Since $[\ \sigma^{(s)}(B_{k-1}), 0, 0\ ]_{T_{k+1}} \in P(F)$, by *Lemma* 1, $[\ \sigma^{(s-1)}(B_{k-1}), \sigma^{(n-1)}(0), \sigma^{(n-1)}(0)\ ]_{T_{k+1}} \in P(F)$.

Case 1: There exist $\alpha \ne 0$, and $\beta \ne 0$, such that $[\ \sigma^{(s)}(\ B_{k-1}\ ), \alpha, \beta\ ]_{T_{k+1}} \in P(F)$.

Case 2: If Case 1 is not true, then for any α and β, $[\ \sigma^{(s)}(\ B_{k-1}\ ), \alpha, \beta\ ]_{T_{k+1}} \in P(F)$ implies that either $\alpha = 0$ or $\beta = 0$.
Proof of Case 1: If there exist $\alpha \neq 0$, and $\beta \neq 0$ such that $[\ \sigma^{(s)}(\ B_{k-1}\ ), \alpha, \beta\ ]_{T_{k+1}} \in P(F)$, then

$$[\ \sigma^{(s-1)}(\ B_{k-1}\ ), \sigma^{(n-1)}(\ \alpha\ ), \sigma^{(n-1)}(\ \beta\ )\ ]_{T_{k+1}} \in P(F)$$

Since $\alpha \neq 0$, and $\beta \neq 0$, so $\sigma^{(n-1)}(\ \alpha\ ) \neq \sigma^{(n-1)}(\ 0\ )$, and $\sigma^{(n-1)}(\ \beta\ ) \neq \sigma^{(n-1)}(\ 0\ )$,

$$F([\ \sigma^{(s-1)}(\ B_{k-1}\ ), \sigma^{(n-1)}(\ 0\ ), \sigma^{(n-1)}(\ 0\ )\ ]_{T_{k+1}},\ [\ \sigma^{(s-1)}(\ B_{k-1}\ ), \sigma^{(n-1)}(\ \alpha\ ), \sigma^{(n-1)}(\ \beta\ )\ ]_{T_{k+1}}\ )$$

$$= [\ \sigma^{(s)}(\ B_{k-1}\ ), \sigma^{(n-1)}(\ 0\ ), \sigma^{(n-1)}(\ 0\ )\ ]_{T_{k+1}} \in P(F).$$

By *Lemma* 1, $[\ \sigma^{(s+1)}(B_{k-1}), 0, 0\ ]_{T_{k+1}} \in P(F)$.

Proof of Case 2: For any $\alpha \neq 0$, and $\beta \neq 0$, we have that $[\ \sigma^{(s)}(\ B_{k-1}\ ), \alpha, 0\ ]_{T_{k+1}} \in P(F)$, and $[\ \sigma^{(s)}(\ B_{k-1}\ ), 0, \beta\ ]_{T_{k+1}} \in P(F)$.

By *Lemma* 1, $[\ \sigma^{(s-1)}(\ B_{k-1}\ ), \sigma^{(n-1)}(\ \alpha\ ), \sigma^{(n-1)}(\ 0\ )\ ]_{T_{k+1}} \in P(F)$, and $[\ \sigma^{(s-1)}(\ B_{k-1}\ ), \sigma^{(n-1)}(\ 0\ ), \sigma^{(n-1)}(\ \beta\ )\ ]_{T_{k+1}} \in P(F)$.

Since $\alpha \neq 0$, and $\beta \neq 0$, so $\sigma^{(n-1)}(\ \alpha\ ) \neq \sigma^{(n-1)}(\ 0\ )$, and $\sigma^{(n-1)}(\ \beta\ ) \neq \sigma^{(n-1)}(\ 0\ )$,

$$F([\ \sigma^{(s-1)}(\ B_{k-1}\ ), \sigma^{(n-1)}(\ \alpha\ ), \sigma^{(n-1)}(\ 0\ )\ ]_{T_{k+1}},\ [\ \sigma^{(s-1)}(\ B_{k-1}\ ), \sigma^{(n-1)}(\ 0\ ), \sigma^{(n-1)}(\ \beta\ )\ ]_{T_{k+1}}\ )$$

$$= [\ \sigma^{(s)}(\ B_{k-1}\ ), \sigma^{(n-1)}(\ 0\ ), \sigma^{(n-1)}(\ 0\ )\ ]_{T_{k+1}} \in P(F).$$

By *Lemma* 1, $[\ \sigma^{(s+1)}(B_{k-1}), 0, 0\ ]_{T_{k+1}} \in P(F)$. □

Lemma 6: For any $B_{k-1} = [\ b_1, b_2, \ldots, b_{k-1}\ ]_{T_{k-1}}$,where $T_{k-1} = \{\ t_1, t_2, \ldots, t_{k-1}\ \} \subset T_{k+1}$, if $[\ \sigma^{(s)}(B_{k-1}), 0, 0\ ]_{T_{k+1}} \in P(F)$, for some $s \in I$, then $[\ B_{k-1}, \alpha, \alpha\ ]_{T_{k+1}} \in P(F)$, for all $\alpha \in I$.
.

Proof: The same as the proof of *Lemma* 6 in the last section. □

Lemma 7: For any $B_{k-1} = [\ b_1, b_2, \ldots, b_{k-1}\ ]_{T_{k-1}}$, and any $\alpha \in I$, we have $[\ B_{k-1}, \alpha, \alpha\ ]_{T_{k+1}} \in P(F)$.
This implies that if $b_i = b_j$ for some $i \neq j$, then $B_{k+1} = [\ b_1, b_2, \ldots b_i, \ldots b_j, \ldots, b_{k+1}\ ]_{T_{k+1}} \in P(F)$.

Proof: The same as the proof of *Lemma* 7 in the last section. □

Proof of $\mathbf{A_{k+1}}$:

Proof: We need to prove that for any $B_k = [\, b_1, \ldots, b_i, \ldots b_j, \ldots, b_k \,]_{T_k}$, and any $\beta = \sigma^{(t)}(0) \in I$, $[\, B_k, \beta \,]_{T_{k+1}} \in P(F)$. By *Lemma* 7, if there exist $i \neq j$, and $b_i = b_j$, where $1 \leq i, j \leq k$, then $[\, B_k, \beta \,]_{T_{k+1}} \in P(F)$.

Now, we discuss the case: All b_i's in B_k are distinct. There exists s such that $[\, B_k, \sigma^{(s)}(0) \,]_{T_{k+1}} \in P(F)$. By *Lemma* 1, $[\, \sigma^{(n-1-s)}(B_k), \sigma^{(n-1)}(0) \,]_{T_{k+1}} \in P(F)$. Since all b_i's in B_k are distinct, there exists a b_i in B_k, such that $\sigma^{(n-1-s)}(b_i) \neq \sigma^{(n-1)}(0)$. By *Lemma* 7, $[\, \sigma^{(n-1-s)}(B_k), \sigma^{(n-1-s)}(b_i) \,]_{T_{k+1}} \in P(F)$.

$$F([\, \sigma^{(n-1-s)}(B_k), \sigma^{(n-1)}(0) \,]_{T_{k+1}}, [\, \sigma^{(n-1-s)}(B_k), \sigma^{(n-1-s)}(b_i) \,]_{T_{k+1}})$$
$$= [\, \sigma^{(n-s)}(B_k), \sigma^{(n-1)}(0) \,]_{T_{k+1}} \in P(F).$$

Since all b_i's in B_k are distinct, there exists a b_i in B_k, such that $\sigma^{(n-s)}(b_i) \neq \sigma^{(n-1)}(0)$. By *Lemma* 7, $[\, \sigma^{(n-s)}(B_k), \sigma^{(n-s)}(b_i) \,]_{T_{k+1}} \in P(F)$.

$$F([\, \sigma^{(n-s)}(B_k), \sigma^{(n-1)}(0) \,]_{T_{k+1}}, [\, \sigma^{(n-s)}(B_k), \sigma^{(n-s)}(b_i) \,]_{T_{k+1}})$$
$$= [\, \sigma^{(n-s+1)}(B_k), \sigma^{(n-1)}(0) \,]_{T_{k+1}} \in P(F).$$

Continue doing this, we have for any r, $[\, \sigma^{(r)}(B_k), \sigma^{(n-1)}(0) \,]_{T_{k+1}} \in P(F)$. We know that $\beta = \sigma^{(t)}(0)$. Let $r = n\text{-}1\text{-}t$. We have that
$[\, \sigma^{(n-1-t)}(B_k), \sigma^{(n-1)}(0) \,]_{T_{k+1}} \in P(F)$. By *Lemma* 1, we have $[\, B_k, \sigma^{(t)}(0) \,]_{T_{k+1}} = [\, B_k, \beta \,]_{T_{k+1}} \in P(F)$. $\mathbf{A_{k+1}}$ is proved. □

Therefore, by induction, A_k is true for all k. Especially, A_{n^2} is true. The function F_1 is functionally complete.

4 Summary

By similar methods, we can prove that other classes of functions $F_2, F_3, \ldots\ldots, F_6$ are functionally complete. The completeness of many classes of functions with the permutation function σ can be examined in this way: A_1, A_2, then by induction A_{n^2}. If a function
F(p, q) is functionally complete, then either this function satisfies the property F (p, p) = σ (p) for some permutation σ, or some composition of function F (p, q) satisfies this property. Therefore, finding more classes of functionally complete functions with function σ is very important in the research of finding all functionally complete functions in the space $\mathbb{C}$.

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