Complementary Subcontinua

Richard Winton, Ph.D. †

Abstract

Continua are sometimes defined as compact, connected metric spaces. In this paper we use the more general definition of a continuum as a compact, connected, Hausdorff topological space. An n-pod is defined to be a subcontinuum of a topological space whose boundary contains exactly n points, where n is an integer greater than 1. Preliminary results of general topological spaces, homogeneous continua, and n-pods are developed to provide access to the main result. Finally, a symmetry is established among n-pods by verifying that for each n-pod in a homogeneous continuum, there exists a complementary n-pod containing the same boundary.

Introduction

In 1980 Forest Wayne Simmons demonstrated the existence of a type of symmetry in homogeneous continua. More specifically, Simmons showed that in a homogeneous continuum, each subcontinuum with two point boundary has a complement whose closure is a subcontinuum with the same boundary [1, Corollary 2, p. 63]. This paper generalizes Simmon's corollary by establishing a similar result for any subcontinuum with finite boundary in a homogeneous continuum.

Definitions

If H is a subset of a topological space X, then Int(H), Cl(H), and Bd(H) are the topological interior, closure, and boundary of H, respectively. A continuum is a compact, connected Hausdorff space. A separation $A|B$ of a space X is a partition of X into nonempty relatively open sets A and B. A subset S of X separates X if and only if X is connected but $X-S$ is not connected. The separation number S(X) of a topological space X is the smallest number of points in X which separates X. If n is an integer greater than 1, then an n-pod of a space X is a subcontinuum of X whose boundary contains precisely n points. Furthermore, n is the pod number of X if and only if X contains an n-pod but X contains no k-pod whenever k is an integer and $1 < k < n$. The pod number of X is denoted by P(X).

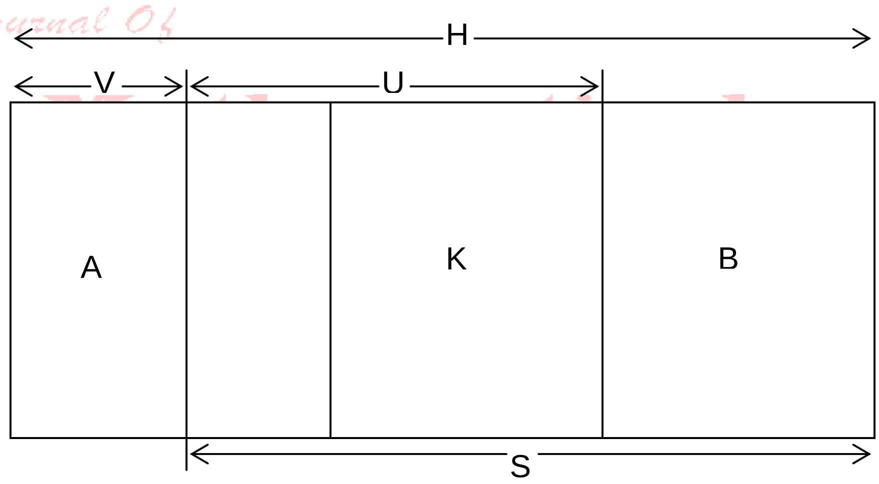
Preliminary Results

When the relative complement $H-K$ of connected subsets of a topological space is not connected, then the union of each component of $H-K$ with K must be connected. We will need a special case of this result, which is formalized in the following lemma.

Lemma 1. Suppose H and K are connected subsets of a topological space X. If $A \mid B$ is a separation of H–K, then $A \cup K$ and $B \cup K$ are connected.

Proof. If $A \cup K$ is not connected, then there is a separation $U \mid V$ of $A \cup K$, so that either $K \subseteq U$ or $K \subseteq V$. Without loss of generality, assume that $K \subseteq U$. Define $S = U \cup B$.

Figure I



Since $K \subseteq U$, then $V \cap K = \varnothing$. Therefore $V \subseteq A$, and so $V \cap H \neq \varnothing$ since $A \subseteq H$. Furthermore, $S \cap H \supseteq B \cap H \neq \varnothing$ since $B \subseteq H$.

Since $V \subseteq A$, then V and B are mutually separated. Hence V and S are mutually separated, and so $V \cap H$ and $S \cap H$ are mutually separated as well.

Finally, $V \cup S \supseteq H$, so that $(V \cap H) \cup (S \cap H) = (V \cup S) \cap H = H$. Thus $(V \cap H) \mid (S \cap H)$ is a separation of H. This is a contradiction since H is connected, and so $A \cup K$ is connected. Similarly, $B \cup K$ is connected.

The separation of a homogeneous continuum by a finite, minimal separating set of cardinality n produces a pair of disjoint open subsets of the space. Furthermore, the union of each of these open subsets with the separating set is an n-pod whose boundary is the separating set itself. We state this fact in the following lemma.

Lemma 2. Suppose X is a homogeneous continuum, S(X) = n, and $S = \{x_i\}_{i=1}^{n} \subseteq X$.

If $A \mid B$ is a separation of X–S, then $A \cup S$ and $B \cup S$ are n-pods in X with common boundary S.

Proof. Since no single point can separate the homogeneous continuum X, then $n \geq 2$. Define $S_1 = S - \{x_1\}$, so that $X - S_1$ is connected since $|S_1| = n-1 < S(X)$. Since $A|B$ is a separation of $X-S = (X - S_1) - \{x_1\}$, then $A \cup \{x_1\}$ is connected by Lemma 1. Similarly $A \cup \{x_i\}$ is connected for $2 \leq i \leq n$, and so $A \cup S$ is connected.

Since B is open in X, then $A \cup S = X-B$ is a closed subset of the compact space X. Hence $A \cup S$ is compact, and is thus a subcontinuum of X.

Finally, $B \cup \{x_i\}$ is connected for $1 \leq i \leq n$ by an argument similar to that above for $A \cup \{x_i\}$. However, if $1 \leq i \leq n$ and $x_i \notin Cl(B)$, then $\{x_i\}|B$ is a separation of $B \cup \{x_i\}$, a contradiction. Therefore $S \subseteq Cl(B)$, and so $B \cup S \subseteq Cl(B)$. On the other hand, if $p \notin B \cup S$, then $p \in A$. Since A is open in X, then $p \notin Cl(B)$, and therefore $Cl(B) \subseteq B \cup S$. Thus $Cl(B) = B \cup S$. Furthermore, since $A \cup S$ is closed, then $Cl(A \cup S) = A \cup S$. Hence $Bd(A \cup S) = Cl(A \cup S) \cap Cl(B) = (A \cup S) \cap (B \cup S) = (A \cap B) \cup S = S$.

Hence $A \cup S$ is an n-pod in X with boundary S. Similarly for $B \cup S$.

Lemma 2 showed that in a homogeneous continuum, minimal separating sets induce n-pods in the space. The following result provides a (weak) converse.

Lemma 3. If H is an n-pod in a connected topological space X, then Bd(H) separates X.

Proof. Suppose $Bd(H) = \{x_i\}_{i=1}^{n}$ $(n > 1)$. Therefore H is infinite, and so $Int(H) = H-Bd(H) \neq \emptyset$. If $X-H = \emptyset$, then $Bd(H) = \emptyset$, a contradiction. Thus $X-H \neq \emptyset$. Clearly $Int(H) \cap (X-H) = \emptyset$. Furthermore, $Int(H) \cup (X-H) = X-Bd(H)$ since H is closed.

Hence {Int(H), X–H} is a partition of X–Bd(H).

Clearly Int(H) is open in X–Bd(H). Since H is closed in X, then X–H is open in X–Bd(H) as well.

Hence $Int(H)|(X-H)$ is a separation of X–Bd(H), and so Bd(H) separates X.

We now have the results necessary to confirm that the pod number and separation number in a homogeneous continuum are the same.

Corollary 4. If X is a homogeneous continuum which contains an n-pod for some integer $n > 1$, then $P(X) = S(X)$.

Proof. Since no single point can separate the homogeneous continuum X, then $S(X) > 1$. Furthermore, since X contains an n-pod whose finite boundary separates X by Lemma 3, then $S(X) < \infty$. Therefore X contains a finite subset S such that $|S| = S(X)$ and S separates X. Hence there exists a separation $A|B$ of X–S, so that $A \cup S$ and $B \cup S$ are S(X)-pods by Lemma 2. Thus $P(X) \leq S(X)$.

Conversely, since X contains a P(X)-pod H, then Bd(H) separates X by Lemma 3. Therefore $S(X) \leq |Bd(H)| = P(X)$. Hence P(X) = S(X).

In view of the result in Corollary 4, Lemma 2 may now be reworded, replacing the separation number S(X) of the homogeneous continuum X with the pod number P(X).

Corollary 5. Suppose X is a homogeneous continuum, P(X) = n, and $S = \{x_i\}_{i=1}^{n} \subseteq X$.
If $A|B$ is a separation of X–S, then $A \cup S$ and $B \cup S$ are n-pods in X with common boundary S.

Proof. Since P(X) = n, then S(X) = n by Corollary 4. Thus by Lemma 2, $A \cup S$ and
$B \cup S$ are n-pods in X with common boundary S.

We are now prepared to present the main result of this paper, which states that in a homogeneous continuum, n-pods occur in “complementary pairs”.

Main Theorem

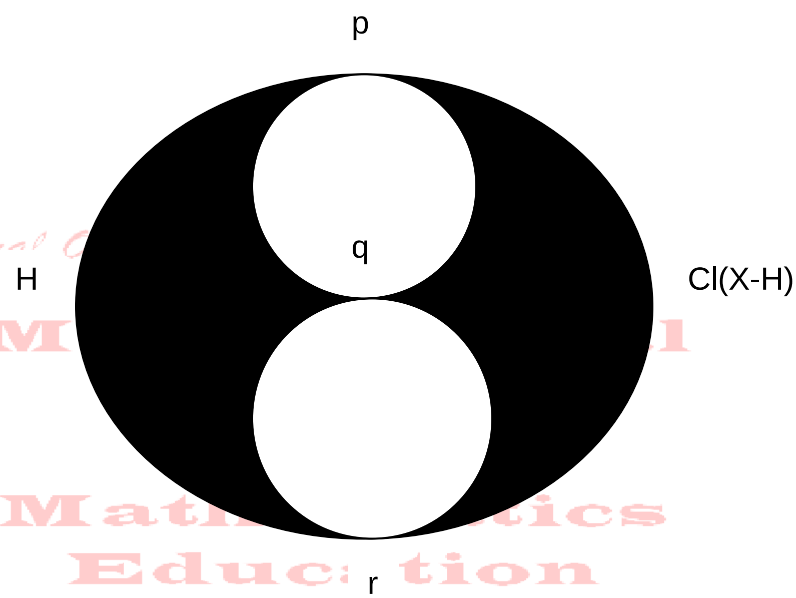
Theorem 6. Suppose X is a homogeneous continuum with P(X) = n. If H is an n-pod in X, then Cl(X–H) is also an n-pod in X. Furthermore, Bd[Cl(X–H)] = Bd(H).

Proof. Clearly Int(H) and X–H are open in X. Since H is infinite, then Int(H) $\neq \emptyset$. Furthermore, if X–H = $\emptyset$, then Bd(H) = $\emptyset$, a contradiction. Thus X–H $\neq \emptyset$. Hence Int(H) and X–H are nonempty open sets in X–S.

Furthermore, Int(H) $\cup$ (X–H) = X–Bd(H) since H is closed. Thus Int(H)|(X–H) is a separation of X–Bd(H). By Corollary 5, Cl(X–H) = (X–H) $\cup$ Bd(H) is an n-pod in X with boundary Bd(H).

Figure II illustrates Theorem 6, where the boundary of H is Bd(H) = {p,q,r}.

Figure II



Conclusion (and definition)

Suppose H is an n-pod in a homogeneous continuum X with P(X) = n. Based on Theorem 6, H and Cl(X–H) will be called complementary n-pods in X.

† *Richard Winton, Ph.D.*, Tarleton State University, USA

References

[1] Simmons, F.W. (1980). When Homogeneous Continua Are Hausdorff Circles (or Yes, We Hausdorff Bananas). *Continua*, *Decompositions*, *and Manifolds*, *University of Texas Press* 62-73.