A Brief Historical Antecedents to the Evolution of Geometry Education

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Abstract

The contributions of geometry to the development and advancement of the other branches of mathematics, physics, astronomy, chemistry, biology, engineering, architecture, and arts are recognized by historians, mathematicians, scientists, artists, and educators. Perhaps, the need for learning geometric concepts and skills and their applications in the 21st century is more crucial than any other time in the history of mankind. We observe the implications of geometric skills and concepts in our every day lives, whether we are in need of a simple estimation such as the number of shingles to replace the roof in our house or a complicated subject such as defining the curvature of space and the path of moving bodies to learn the laws of universe.

Utilizing geometric concepts and skills in learning and problem solving in other scientific disciplines require an efficient level of proficiency in a large number of geometric concepts or skills. Considering the importance of geometry in school curriculum, an effective method of teaching and learning geometry is desirable for mathematics educators. A brief review of geometry textbooks published in the United States between 1854 and 2001 helps us to explore and examine the instructional design and methods utilized for teaching and learning of this discipline.

Background

From 8000 B.C. (the beginning of the development of agriculture) when Sumerians used simple plain clay tokens in the shapes of disks, spheres, cylinders, and cones to keep track of their different kinds of goods (before the invention of writing) to the 20th century when Einstein's new geometry of space and time attempted to explain the natural motions of the objects in the universe, geometry has helped mankind to organize, classify, describe, represent, explain, interpret, define, and express in a logical way the world surrounding us. Kline (1963) emphasizes the importance of geometry when he writes:

> The story is told that the Greek philosopher Aristippus and some friends were shipwrecked on what appeared to be a deserted island near Rhodes. The company was downcast at its ill fortune when Aristippus noticed some geometric diagrams drawn on the beach sand. He told his companions: Be of good cheer, I see traces of civilized man. (p. 83).

The study of geometry is an essential part of the grades K-12 curriculum. The National Council of Teachers of Mathematics, NCTM, (1989) for grades K-4 recommends topics in geometry and spatial sense such as describing, modeling, drawing, classifying, combining, dividing, and changing shapes. For grades 5-8, NCTM suggests identifying and comparing geometric figures in one, two, and three dimensions and the applications of geometric properties and relationships in problem solving and real life situations. NCTM also recommends, for grades 9-12, such topics in geometry as representing problem situations with geometric models, classifying figures in terms of congruence and similarity, and investigating and understanding the axiomatic system.

Historical Overview of the Development of Geometry

The history of geometry is as old as the history of man. To explore the evolution of geometry and the formation of the geometry curriculum, it is essential to state a brief history of the development of geometry. One cannot overlook the fact that as educators what we teach, in any discipline, is nothing but the history of that particular discipline. Zebrowski (1999) states,

> We teach what has been learned in the past, hoping that the universe will cooperate in allowing our students to apply this past knowledge to the unfolding future. This is as true in music and art as it is in science and mathematics. We could not teach, nor could anyone learn, if we did not believe that the universe displays a historic continuity. (p. x).

Great teachers through the ages have always been acutely aware of the genetic principle and its implications in education. Jones (1969) states, "There were some inspired teachers, such as Ernst Mach, who in order to explain an idea referred to its genesis and retraced the historical formation of the idea" (p. 3). Exploring the history of geometry indicates the following three distinct periods: Intuitive, Classical, and Modern.

1.) Intuitive geometry (8000 B.C. - 500 B.C.): archaeological evidence suggests, was born in the Middle East at the time of Sumerians, and was further developed by Babylonians, and then Egyptians. Rouse Ball (1960) states,

> The periodical inundation of the Nile which swept away the landmarks in the valley of the river, and by altering its course increased or decreased the taxable value of the adjoining lands, rendered a tolerably accurate system of surveying indispensable, and thus led to a systematic study of the subject by the priests. (p. 5).

Smith (1953) writes that the earliest geometry "was intuitive in its nature; that is, it sought facts relating to mensuration without attempting to demonstrate these facts by any process of deductive reasoning" (p. 270). One might say that learning the skills of intuitive geometry could continue to thrive without any formal education because it was only through the experience and performance of related professions that intuitive geometry was transferred from one generation to another during the first period of the history of geometry.

The first important surviving manuscript about geometry is a document written by an Egyptian priest around 1700 B.C. This manuscript called Ahmes consists of arithmetic and mensuration, areas of rectangles and circles.

2.) Classical geometry (500 B.C.-1600 A.D.): it was the result of systemization and formalization of the intuitive geometry and mathematical concepts and ideas. Stwertka (1987) points out,

> The Greek philosopher Herodotus claimed that geometry was [a] gift of the Nile and indeed its beginning was most certainly in the early Babylonian and Egyptian civilizations. The Greek mathematicians were less concerned with applying geometry to practical problems than with teaching abstract reasoning and contemplating the ideal and beautiful. There was also the hope that mathematics could be a means of understanding the universe. Philosophers such as Thales, Pythagoras, Eudoxus, and Euclid produced an amazing amount of first class mathematics. The work of many schools of thought was unified by Euclid, whose mathematical creations and insights were to dominate geometry for the next 2000 years…the rules of geometry were first systematically stated by Euclid around 300 B.C. (p. 9).

The Greeks established geometric facts based on deductive reasoning not empirical procedures. Thales is credited with being the first mathematician who used deductive reasoning in geometry in the first half of the sixth century B.C. Later, in the second half of the sixth century B.C., Pythagoras attempted to incorporate deductive reasoning in order to systemize geometry. It is reputed that Hippocrates was the first to extend the earlier works of Greek mathematicians by using a few definitions and assumptions to form chains of propositions. These assumptions have become known as "axioms" and/or "postulates". Theudius and others followed Hippocrates by further developing a logical presentation of geometry. Finally, Euclid organized and presented the achievements of the Greek mathematicians in his book called the "*Elements*". It is also essential to mention the contributions of other philosophers and mathematicians such as Leon, Anaxagoras, Plato, Hypsicles, Apollonius, Hero, Archimedes, Menelaus, Ptolemy, Pappus, and Diophantus to the development of the geometry of this period.

During the period of the Dark Ages that began with the fall of the Roman Empire and extended to the 11th century, the development of geometry like any other branch of mathematics or science was interrupted in Europe. With the rise of the Moslem Empire and its interest in mathematics and science during the seventh century the Greek method of the logical presentation of geometry was preserved. Greek books of mathematics and science were translated into Arabic. During this period, the Indian mathematician Brahmagupta, the Arab mathematician Abulwefa and the Persian mathematician and astronomer Khayyam made some significant contributions to the development of geometry.

The rebirth of learning in Europe in the 11th century called for a translation of the Greek classics that had been preserved outside Europe. Greek classics in science and mathematics such as the "*Elements*" were retranslated into Latin. This translation of scholarly work continued during the Renaissance to the 17th century.

3.) Modern geometry (after 1750 A.D.): Eves (1965) states that some sources place the beginnings of modern geometry with the work of Saccheri and Lambert as they continued to attempt to prove Euclid's fifth postulate from the first four (p. 56). Euclid's fifth postulate states,

> If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles". (Robold, 1969, p. 208).

A logically equivalent version of this postulate is as follows: there is exactly one line parallel to a line through a given point not on the line (Playfair's postulate). Eves (1969), concerning Euclid's fifth postulate, states,

> This postulate lacks the terseness of Euclid's other postulates, and it does not seem to possess the quality, demanded by Greek material axiomatics, of self-evidence or ready acceptability on the part of the reader.... It was natural to wonder if the postulate was really needed at all, and to think that perhaps it could be derived as a theorem from the remaining postulates, or, at least, it could be replaced by a more acceptable equivalent…. The attempts to derive the parallel postulate as a theorem from remaining postulates of Euclid's *Elements* occupied geometers for over two thousand years. (p. 184)

Playfair attempted to substitute his postulate for Euclid's fifth postulate but did not succeed. Legendre attempted to prove the parallel postulate as a theorem. Gauss, Bolyai, Lobachevsky, Beltrami, Klein, Poncelet, Poincare, and Riemann contributed to the development of non-Euclidean geometry by creating models of geometries that obeyed every Euclidean postulate except the fifth. The negation of the fifth postulate in these models showed the independence of Euclid's fifth postulate from the first four.

Besides the difficulties surrounding the parallel postulate there are some other logical imperfections with Euclid's axiomatic system. Euclid established five postulates and five axioms for geometry. Although these initial premises justify the proof of a large number of the propositions in geometry, but there are some propositions that cannot be drawn from the Euclid's first principles. Some additional postulates are needed. For example, the tacit assumption that the straight line is of infinite extent is such a proposition and was used by Euclid. Eve (1965) states,

> In short, the truth of the matter is that Euclid's first principles are simply not sufficient for the derivation of all of the 465 propositions of the *Elements*. In particular, the set of postulates needs to be considerably amplified. The work of perfecting Euclid's initial assumptions, so that all of his geometry can rigorously follow, occupied mathematicians for more than two thousand years…. Not

> only is Euclid's work marred by numerous tacit assumptions, but some of his preliminary definitions are also open to criticism…. In Euclid's development of geometry the terms *point* and *line*, for example, could well have been included in a set of primitive terms for the discourse. Euclid's definition of a point as "that which has no part" and of a line as "length without breadth" are easily seen to be circular and therefore, from a logical point of view, woefully inadequate. (p. 40).

The complete postulational development of Euclidean geometry was accomplished by mathematicians such as Pasch (1843 – 1930), Peano (1858 - 1932), Pieri (1860 – 1904), Hilbert (1862 – 1943), and Birkhoff (1884 – 1944). The Hilbert's modern axiomatic development of geometry consists of 21 postulates. Hilbert's set of postulates has been incorporated in the instructional content of modern high school geometry textbooks in the United States.

Historical Background in Instructional Delivery of Geometry

In 1894 the "Report of the Committee of Ten" in reference to geometry stated,

> It is the belief of [the] conference that the course here suggested, if skillfully taught, will not only be of great educational value to all children, but will also be a most desirable preparation for later mathematical work. Then, too, while it will on one side supplement and aid the work in arithmetic, it will on the other side fit in with and help the elementary instruction in physics, if such instruction is to be given. (NCTM, 1933, p.97).

A brief review of the history of the development of geometry demonstrates that contrary to some misconceptions, geometry is not a static subject. Rather, it is a relatively dynamic discipline. The evolution of geometry from Euclidean geometry to modern geometry is one of the most dynamic developments and advancements in mathematics. The methods of teaching geometry as well as the strategies of learning geometry are as dynamic as the evolution of geometry. To explore the most effective method of learning plane geometry it is advantageous to examine the different strategies through which, historically, the discipline has been taught, practiced, learned, and retained.

The Instructional Design and Methods of Geometry Textbooks in the United States

To explore, historically, the methods of teaching and learning plane geometry in the United States, we examined the instructional design and methods of geometry textbooks published in the United States between 1854 and 1999. This examination focused on the methods and the processes utilized for the reinforcement of geometry content.

It seems Thomas Hills wrote the first geometry textbook, in the United States, in 1854 for the learners between six to twelve years old. The book was titled "*First Lessons in Geometry*". Thomas Hills (1854), as cited in NCTM (1933), in the preface of his book claims,

> Many parts of this book will, however, be found adapted, not only to children, but [also] to pupils of adult age. I have tried to present them in simple and attractive dress. I have addressed the child's imagination, rather than his reason, because I wish to teach him to conceive of forms. The child's power of sensation is developed, before his power of conception, and these before his reasoning powers… I have, therefore, avoided reasoning, and simply given interesting geometrical facts, fitted, I hope, to arouse a child to the observation of phenomena, and to [the] perception of forms as real entities. (p. 63).

Brooks, E. authored "*The Normal Elementary Geometry: Embracing a Brief Treaties on Mensuration and Trigonometry*" in 1884. The book was recommended for academies, seminaries, high schools, normal schools, and advanced classes in common school. It is 191 pages in length and consisted of eight "books" (chapters). The first two chapters of the book are devoted to definitions of geometry and a review of the algebraic concepts essential in learning geometry. In chapters three through eight, the author discusses a few geometric concepts or skills in each chapter and presents between five to twenty two exercises to enhance the students learning. The recommended problems (practical exercises) at the end of each chapter contain only the concepts or skills discussed in that particular chapter. There are approximately 325 exercises in the book and students are provided with the answers to a majority of these problems. Brooks (1884) suggests,

> The practical exercises should be solved by all classes. The easier problems may be assigned in connection with the theorems [,] which they illustrate; the others may be deferred until the book [chapter] upon which they depend is completed. The most difficult problems may be omitted until the whole geometry is completed. (p. 12).

Wells, W. authored *"The Essentials of Geometry (Plane)"* in 1898. The book is 232 pages and composed of five different chapters. Each chapter contains several sections. In each section, the author defines a few geometric terms, introduces a new concept, works a few example problems, and presents some problems for reinforcement. There are approximately 486 problems in this textbook. The author provides a number of hints to some of the problems. In addition, the author provides answers to 225 of these problems. A very limited number of drawings and figures are employed to explain the geometric concepts and skills discussed in the textbook.

Wentworth, G. wrote "*Plane and Solid Geometry*" in 1899. The book is 473 pages and consists of an introduction and nine different books (chapters). The book offers 910 problems for learners to practice. There are between seven to ten problems at the end of the each section of each chapter to reinforce the learning of the concepts and skills taught in the same section. In addition, at the end of each chapter there are a large number of exercise problems reinforcing only the material discussed in that particular chapter. There are very few illustrations to explain the concepts or the problems in the book. Wentworth (1899) in a note to the instructors of his book writes,

> He [the learner] should be encouraged, in reviewing each book [chapter], to do the original exercises; to state the converse propositions, and determine whether they are true or false; and also to give well-considered answers to questions [,] which may be asked him on many propositions. (p. v).

Bruce, W.H. wrote *"Elements of Plane Geometry"* in 1929. The 278-page book consists of an introduction and five chapters. Each chapter consists of a few sections. In each section, the author defines a few terms, discusses some concepts, introduces a few problems containing these concepts, and assigns some problems concerning the concepts and skills discussed in that particular chapter to the students. There are 555 problems in the book. The author does not provide the students with any hint, answer, or solution to these problems and uses a very few drawings and figures in the instructional design of the book. Bruce (1929) states,

> Throughout the study of the subject the student should be made to recognize that geometry is pure logic; that is pre-eminently that type of reasoning that leads from a few assumptions, axioms, to conclusions invariable and immutable. The subject matter of geometry is wholly independent of commensurable quantities-those that may be measured exactly in terms of any notion or expressed in concrete numbers of any scale whatsoever. (p. iii).

Strader, W. and Rhoades, L. authored "*Plane Geometry: A Modern Text*" in 1934. The book is 399 pages long and consists of an introduction, six chapters and a supplement part. The instructional design of the book follows the same format as the prior textbooks in geometry. Different geometric concepts or skills are introduced to the students in each chapter. At the end of each section of each chapter the authors presents two sets of exercises. The first set is the oral exercises that usually consist of about ten questions concerning the definitions that the learners were exposed to in that section. The second set of assignment is about ten to twenty written exercises to practice the skills discussed only in that particular section. If the students are presented with more than one concept or skill in a single section or chapter, the authors separate those exercises from one another by their particular concepts or skills such as: "(exercises, parallel lines), (exercises, congruent triangles), (exercises, parallelograms), (exercises, angles of polygons)". There are about 2929 exercises in the textbook. There are 77 problems concerning the applications of geometry in physics, mechanics, construction, and engineering. The Authors do not provide any answer, hint, or solution to the exercises in this text. There are very few illustrations and pictures to explain the concepts as well as the problems. In a rather lengthy introduction (38 pages!) the authors introduce the students to the definition of terms used in their book and also indirectly expose them to the process of deductive reasoning. Strader and Rhoads (1934) state,

> You will observe as you progress in geometry that your work is somewhat like that of a builder. You will lay the foundation for the structure of geometry by defining certain fundamental terms and by accepting some general statements as truths. Any new material, which enters into the construction of the building, is tested and approved, and in like manner each statement you use in geometry must be proved. The proofs piece together facts [,] which are already known and this means show new statements to be true… As you study about figures, their properties, and uses, you will gain not only knowledge, but [also] greater power in correct thinking. (p. 2).

Keniston, R. and Tully, J. authored *"Plane Geometry"* in 1946. This 391 pages book is composed of an introduction and sixteen chapters. Each chapter contains a few sections. The authors introduce a few geometric concept or skill in each section, work some relevant problems, and present a few problems to reinforce the materials discussed in that particular section to the learners. There are some review problems at the end of each chapter concerning the concepts and the skills discussed in different sections of that specific chapter. There are about 2355 problems in the book. There are about 590 problems with real life applications. The authors do not provide any hint, answer, or solution for any of the problems. The authors use a large number of drawings, figures, illustrations, and photographs to explain and discuss the geometric concepts and skills. Keniston and Tully (1946) state,

> We offer this text in plane geometry as one [,] which retains the best of the conventional course, and at the same time [,] complies with the new trends in education.
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> The general format, the many attractive illustrations, and the informal style mark the book as modern, and help to make the subject matter interesting. The detailed explanations, the emphasis on an awareness of geometry as it permeates all the areas of knowledge and the use of geometry as a means of stimulating logical and scientific thinking make this a text which contributes to general education, as well as to a professional preparation.... Our first objective is to point out the geometric aspects of environment, thus fostering an appreciation of geometric form and supplying an interesting motive.... Our second objective is to develop an awareness of the logic of geometry and to correlate this with reasoning in nonmathematical thinking.... Our third objective is to encourage independence of thought.... Our fourth objective is to enable the pupil to acquire a thorough knowledge of the facts of geometry needed for continued work in mathematics and allied subjects.... Our fifth objective is to offer a course suitable for pupils of varying needs and abilities. (p. v).

Rosskopf, M. et al authored *"Modern Mathematics, Geometry"* in 1966. The book is 566 pages long and contains 15 chapters. Each chapter consists of a few sections. The authors, in each section, define the basic terms, discuss the new concepts and skills, work a few example problems, and assign a number of problems to reinforce the newly learned concepts and skills. The authors end each chapter with a summary section, a review exercises section, and a "for the adventurous" section. The section titled "for the adventurous" contains a small number of challenging problems. The authors use a sufficient number of illustrations, drawings, figures, and a large number of color photographs to explain the geometric concepts and skills discussed in their book. Rosskopf, et al. (1966) state,

> [*Modern Mathematics, Geometry*] reflects the spirit of the present mathematics curriculum by emphasizing active student involvement, by building and maintaining skills, and by focusing on such unifying themes as structure, sets, number, and proof.... The material of the text develops understanding, builds skills in analysis and proofs, and provides an algebraic treatment of geometry that promises to lead students to real mathematical competency. (P. i).

Jurgensen R., et al. authored *"Modern School Mathematics, Geometry"* in 1969. The 660-page book contains sixteen chapters. The reinforcement in each chapter is composed of oral and written exercises, chapter review problems, and chapter test. There are about 7728 oral and written exercises. The authors use a significant number of drawing, illustrations, graphs, and pictures to facilitate the students' learning of geometry. The material presented in the first three chapters of the book which discuss the language of sets, relationship between sets, induction - a method of discovery, deduction - a method of proof, principles of logic, and mathematical systems reflects the demand for including logical rigor in 1960's and 1970's. Jurgensen, et al. (1969) state,

> In order to meet the demand of industry, the professions, and mathematics itself, mathematicians have been called upon to extend and combine old branches of mathematics as well as to develop new ones. While geometry is one of the oldest branches of mathematics, it is, today, finding new areas of application in such fields as space exploration and rocket design. (p. 1).

Presser, R. and Ringenberg, L. wrote *"Geometry"* in 1971. The authors in each section of this 707-page book, define the geometric terms used in that particular section, discuss a number of concepts or skills, and present some review problems to the learners. There are about 1065 problems in the textbook. The authors provide no answer, hint, or solution to any of the problems. The authors use very few figures and illustrations to explain the concepts and skills. Presser and Ringenberg state,

> Geometry is an important subject because it is practical and useful and at the same times abstract and theoretical. There are two main objectives in this geometry textbook. One is to help students learn a

> body of important facts about geometrical figures. These facts, interpreted physically, are facts about the space in which we live. These facts are important for intelligent citizenship and for success in many careers. The other main objective is to help students attain a degree of mathematical maturity. (p. viii).

Moise, E. and Downs Jr., F. wrote "*Geometry"*, in 1971. The book is 676 pages and contains 19 chapters. Each chapter is composed of two to ten different sections. The instructional design of this book follows the same general path of the previous geometry books. In each section the authors define a few terms, introduce a few new concepts, work a small number of example problems, and present to the learners a relatively large number of problems for reinforcement. There are about 2415 problems in the textbook. The authors do not provide any hint, answer, or solution to the problems. The authors employ a sufficient number of figures to explain the concepts and skills; but they do not use any drawings, diagrams, pictures, or photos in their instructional design. The mathematical content of the book extends the work on topics such as the following: common sense and exact reasoning, sets, real numbers, trigonometry, transformations, and vectors.

Jurgensen, R. et al authored *"Modern Basic Geometry"* in 1973. The 481-pages book contains twelve chapters. Each chapter consists of a few different sections. The authors in each section introduce a new concept, provide some example problem, and assign a large number of problems as exercise. There are approximately 4200 problems. The authors provide answers to a very small portion of these problems (approximately 98 of the problems). A very limited number of drawing and illustration are used to explain the geometric concepts and skills. Jurgensen, et al. (1973) state,

> [*Modern Basic Geometry*] presents geometry at a level [,] which will permit all students to experience success and to enjoy a feeling of achievement. The type of material in the text, the amount of material, and the manner in which it is presented have been planned so that the average student will have a rewarding experience in the geometry course. (p.T1).

Rhoad, R. et al. wrote "*Geometry for Enjoyment and Challenge*" in 1981. This 756-page textbook contains sixteen chapters. Each chapter contains a few sections. The authors begin each section by defining the terms used in that section. Then, they introduce a few new concepts to the students and expose them to some example problems. There are about fifteen problems at the end of each section to reinforce the learning of the new concepts discussed in that particular section. At the end of each chapter learners are assigned about twenty to thirty review problems. The contents of these problems are composed of all the skills discussed in the entire chapter. The textbook contains about 2204 different problems. The authors provide answers to about 1100 of these problems. The authors use diagrams, figures, and tables to explain and clarify the concepts and the problems discussed in the text. Rhoad, et al. (1981) state, "The text is written to be read daily by the student. Acuity in reading, like mastery of the material, should come to students over time. The text has built-in repetition so students gradually develop and continually maintain these skills" (p.1).

Weeks, A. and Adkins, J. published their book titled "*A Course in Geometry: Plane and solid*" in 1982. The 552-page book consists of 24 chapters. The authors, after discussing the concepts and skills in each section, present the learners with about 10 to 20 problems concerning the skills discussed in that section. At the end of each chapter there are review problems that cover all the skills discussed in the entire chapter. There are 3130 problems in the textbook but no solution, hint, or answer is provided to these problems. The authors use relatively limited number of figures and illustration to explain the concepts and problems. Weeks and Adkins (1982) state,

> In order that students may discover many of the results for themselves, exercises which anticipate the proof of a theorem precede the new theorem whenever possible. In the later chapter proofs of theorems of straightforward nature are left to the students. (p. v).

Cummins, J. et al. wrote "*Informal Geometry*" in 1988. The text is 566 pages and is composed of 18 chapters. Each chapter contains about six different sections. Each section of each chapter exposes the students to the new skills and presents them with the exercises designed for that section. There are some review problems at the end of the each chapter that contain all the material discussed in that chapter. There are approximately 3551 problems in this textbook. The authors provide answers for about 1770 of these problems. The text contains figures, diagrams and pictures to clarify the concepts and also the problems. Cummins, et al. (1988) describe, " Every chapter contains a chapter review coordinated to each lesson. Each chapter has at least one 'sharpen your skill' feature that reviews the concepts learned thus far in the chapter or previous arithmetic and algebra skills needed for future lessons" (p. iii).

Serra, M. authored "*Discovering Geometry: An Inductive Approach*" in 1989. The 755-page book contains 15 chapters and each chapter is composed of several different sections. At the end of each section there are exercises designed for that particular section. At the end of each chapter learners are presented with approximately 30 review problems to reinforce the skills learned in that chapter and also the previous chapters. There are about 2487 problems in the text; and author provides hints and solutions to approximately 513 of these problems. There a large number of figures, illustrations, tables, diagrams, pictures and colors to explain and clarify the concepts as well as the problems in this textbook.

Boyd, C., et al. wrote "*Geometry: Integration, Applications, Connection*" in 1998. The 908-page book is composed of 13 chapters. Each chapter contains a few sections, and each section discusses several different definitions, concepts, and skills. The authors expose the learner to the contents of each section, present a few example problems and assign up to 40 problems concerning that section to the students. There are a large number of problems as review problems at the end of each chapter that are designed to reinforce all the concepts and skilled learned in that chapter as well as the previous chapters. There are approximately 4474 problems in this textbook and the authors provide answers to about 2018 of these problems. The authors use figures, illustrations, tables, diagrams, photographs and variety of colors to explain and clarify the concepts and problems.

The review of these textbooks was not intended to be comprehensive, but rather to observe and compare the instructional design and methods of these textbooks with respect to the techniques of learning geometry. The review indicates the following changes in the instructional design and methods of the geometry textbooks published between 1854 and 2000:

1.) The number of pages has increased considerably.
2.) The number of problems in the text has increased.
3.) The number of illustrations, figures, tables, and photographs in the textbooks has increased.
4.) The authors use more and more variety of colors to explain the concepts and problems.
5.) The review problems at the end of the chapters consist of all the concepts and skills discussed in all the sections of that chapter.
6.) The number of topics and concepts discussed in the textbook has increased.
7.) The authors recommend the use of technologies such as graphing calculators and computers.
8.) The number of real life and application problems has increased.
9.) The authors provide answers, hints, and solutions for a number of practice problems.

In most cases, in geometry textbooks published after mid 1900s, the review sections at the end of each chapter represent not only the contents of that particular chapter but also contain some definitions and skills learned in the previous chapters. Providing answers, hints, and in some instances solutions to some of the exercises is a part of the instructional design in the majority of the geometry textbooks published in mid and late 20th century.

Guiding the learners to discover the concepts and reinforce the geometric skills, and also introducing a large number of problems concerning the applications of these concepts and skills are attempts to assist the learner to construct a knowledge system. The success or failure of these changes and improvements in the instructional design of the geometry textbooks depends on instructors' enforcement of these strategies in classrooms.

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