Cancellation and Zero Divisors in Rings

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Abstract

Relationships between cancellation properties and zero divisors in rings are routinely found in textbooks on abstract algebra. However, the results presented in these texts are incomplete or vague in one form or another. Results are sometimes stated in a conditional rather than biconditional form, omitting a converse. Other times partial results are stated for rings with special properties like commutativity, an identity, or both. In other cases results are stated for one type of cancellation (left or right) while omitting the other type. Other sources state the results of this paper in a vague or misleading manner which implies that both left and right cancellation are required to guarantee a connection with the zero divisors of the ring. In this paper we will show that the individual one-sided cancellation laws and the absence of individual one-sided zero divisors are all equivalent. The consequence of this observation is that in an arbitrary ring, left and right cancellation are equivalent.

Introduction

Students who are new to the study of algebraic structures sometimes have a misconception about non-commutativity. In general, they believe that non-commutative structures which contain “one-sided” properties (left identites, left inverses, etc.) are never guaranteed the corresponding “other-sided” properties (right identites, right inverses, etc.). However, a careful study of the definition of a group would suggest that such may not actually be the case. For even a non-abelian group G wil contain an identity element e such that both xe = x and ex = x for each element $x \in G$. Furthermore, each element $x \in G$ has an inverse $x^{-1} \in G$ such that both $xx^{-1} = e$ and $x^{-1}x = e$.

Cancellation in algebraic structures is a desirable property. In structures such as groups, in which each element has an inverse, both left and right cancellation clearly exist. However, in structures which do not guarantee the existence of inverses, cancellation is not assured. Furthermore, in non-commutative structures with no guarantee of inverses, left cancellation is generally considered to be independent of right cancellation. This paper will show that in an arbitrary (non-commutative) ring, left cancellation and right cancellation are equivalent.

Preliminary Results

We begin with the definition of zero divisors. We use the standard definitions for left and right zero divisors. There is apparently some variation in the definition of the zero divisor. Most texts define a zero divisor as either a left or right zero divisor ([1, p. 78], [2, p. 207], [3, p. 170], [5, p. 251], [6, p. 177], [7, p. 228], [8, p. 215], [9, p. 194], [11, p. 61], [13, p. 275], [14, p. 53], [15, p. 248], [16, p. 153], [17, p. 91]). Some texts, however, define a zero divisor to be both a left *and* right zero divisor ([10, p. 116], [12, p. 141]). Most texts also require a zero divisor to be nonzero itself; a few do not ([12, p. 141], [14, p. 53]). The primary result of this paper does not depend on whether a zero divisor must be both a left and right zero divisor or either one. The result does, however, rely on a zero divisor being a nonzero ring element. Thus we will use the following definition for zero divisor.

Definition 1. Suppose R is a ring and r is a nonzero element of R. Then r is a *left zero divisor* of R if and only if R contains a nonzero element s such that rs = 0. Similarly, r is a *right zero divisor* of R if and only if R contains a nonzero element s such that sr = 0. A zero divisor of R is an element of R which is either a left or right zero divisor (or both).

The following result is commonly known, and is included here for completeness. It simply establishes that none of the three types of zero divisors defined above can exist in the absence of the others.

Lemma 2. If R is a ring, then the following are equivalent:

(a) R contains a left zero divisor.
(b) R contains a right zero divisor.
(c) R contains a zero divisor.

Proof: To show (a) implies (b), suppose r is a left zero divisor in R. Then $r \neq 0$ and R contains an element $s \neq 0$ such that rs = 0 by Definition 1. Therefore s is a right zero divisor of R by Definition 1.

To show (b) implies (c), suppose s is a right zero divisor of R. Then by Definition 1 s is a zero divisor of R.

To show (c) implies (a), suppose t is a zero divisor of R. Then t is either a left zero divisor of R or t is a right zero divisor of R by Definition 1. If t is a left zero divisor of R, then the proof is complete. On the other hand, if t is a right zero divisor of R, then by Definition 1 $t \neq 0$ and there exists some $r \in R$ such that $r \neq 0$ and rt = 0. Thus r is a left zero divisor of R, which completes the proof.

The main result of this paper will depend directly on Lemma 2 in the contrapositive form. Thus we restate Lemma 2 as follows.

> If R is a ring, then the following are equivalent:
> (a) R contains no left zero divisors.
> (b) R contains no right zero divisors.
> (c) R contains no zero divisors.

In view of Definition 1 and Lemma 2, the existence of left, right, or two-sided zero divisors in a ring may be restricted to just zero divisors. For if a ring contains one of them, then by Lemma 2 it contains all of them. Furthermore, if a ring fails to contain one of them, then it fails to contain any of them.

It is important to interpret Lemma 2 correctly. It does not conclude that an element r being a left zero divisor, a right zero divisor, and a zero divisor are equivalent. Rather it concludes that the *existence* of a left zero divisor, a right zero divisor, and a zero divisor are equivalent.

We now define the cancellation properties for rings. In particular, a left and right cancellation property is defined. These are the standard definitions which may be found in most texts.

Definition 3. A ring R is said to have the left cancellation property if and only if whenever $r,s,t \in R$, $r \neq 0$, and $rs = rt$, then $s = t$. Similarly, R has the right cancellation property if and only if whenever $r,s,t \in R$, $r \neq 0$, and $sr = tr$, then $s = t$

Since a ring forms an abelian group relative to addition, it will always have the left and right cancellation properties with respect to addition. Therefore, the phrase "cancellation properties" applied to rings will always mean cancellation relative to the ring multiplication.

We proceed now to develop a connection between the cancellation properties and zero divisors. The following lemma is the final result necessary to establish the main theorem of the paper. In this lemma we will show that the absence of either type of one-sided zero divisor in a ring is equivalent to the existence of the corresponding one-sided cancellation property.

Results similar to the following lemma can be found in many texts. However, the results are presented in a weak, incomplete, or non-specific manner. For example, some texts state that in a commutative ring R, if both cancellation laws hold then R has no left or right zero divisors [9, p. 200]. Others present something of a converse, stating that in a commutative ring R with identity, if R has no left or right zero divisors, then both cancellation laws hold ([7, p. 229], [12, p. 143]). In both cases, only a partial result is provided under unnecessarily restrictive conditions. Furthermore, the results above imply a *collective* relationship which shows no independence between left and right cancellation or the absence of left or right zero divisors. The first suggests that both cancellation laws must hold to have any guarantee about the absence of left or right zero divisors. The other implies that the ring must be free of both left and right zero divisors to have any guarantee about cancellation laws. Some texts omit the unnecessary restrictions on the ring. These state that if a ring has no left or right zero divisors, then both cancellation laws hold ([1, p. 83], [13, p. 276]). However, these still provide a partial result while addressing the relationship between cancellation and zero divisors collectively. Still other texts offer a biconditional result between cancellation and zero divisors in *commutative* rings, but also adopt a collective approach [4, p. 5]. Finally, many texts provide the biconditional result between cancellation and zero divisors while omitting the requirement of commutativity, but insist on describing the relationship collectively ([3, p. 171], [5, p. 251], [8, p. 216], [10, p. 116], [15, p. 248], [16, p. 156]). The goal here is to be very specific about the connection between the *left* cancellation property and *left* zero divisors, as well as between the *right* cancellation property and *right* zero divisors, independent of one another. Furthermore, these relationships are established in the context of an *arbitrary* ring.

Lemma 4. Suppose R is a ring.

(a) R has the left cancellation property if and only if R contains no left zero divisors.

(b) R has the right cancellation property if and only if R contains no right zero divisors.

Proof: (a) Suppose R has the left cancellation property. If R contains a left zero divisor r, then $r \neq 0$ and there exists some s such that $0 \neq s \in R$ and $rs = 0$. Therefore $rs = 0 = r0$, so that $s = 0$ by left cancellation. However, this is a contradiction since $s \neq 0$. Hence R contains no left zero divisors.

Conversely, suppose R contains no left zero divisors. If $r,s,t \in R$, $r \neq 0$, and $rs = rt$, then $rs - rt = 0$, and so $r(s-t) = 0$. Since $r \neq 0$ and R contains no left zero divisors, then $s-t = 0$, and so $s = t$. Hence R has the left cancellation property.

(b) Suppose R has the right cancellation property. If R contains a right zero divisor r, then $r \neq 0$ and there exists some s such that $0 \neq s \in R$ and $sr = 0$. Therefore $sr = 0 = 0r$, so that $s = 0$ by right cancellation. However, this is a contradiction since $s \neq 0$. Hence R contains no right zero divisors.

Conversely, suppose R contains no right zero divisors. If $r,s,t \in R$, $r \neq 0$, and $sr = tr$, then $sr - tr = 0$, and so $(s-t)r = 0$. Since $r \neq 0$ and R contains no right zero divisors, then $s-t = 0$, and so $s = t$. Hence R has the right cancellation property.

Main Result

We are now prepared to present the main result of this paper. Specifically, we now show that, in an arbitrary (possibly non-commutative) ring, the existence of left cancellation is equivalent to the existence of right cancellation.

Theorem 5. In a ring R, the following are equivalent:

(a) R has the left cancellation property.

(b) R contains no left zero divisors.

(c) R contains no zero divisors.

(d) R contains no right zero divisors.

(e) R has the right cancellation property.

Proof: The equivalence of (a) and (b) was established in Lemma 4 part (a). The equivalence of (b), (c), and (d) was established in Lemma 2. The equivalence of (d) and (e) was established in Lemma 4 part (b).

Concluding Remarks

As mentioned above, the definition of a zero divisor varies somewhat. It is worth noting that the main result involving the equivalence of left and right cancellation in rings is independent of which definition is used. This issue with the general zero divisor can be avoided by omitting it from Definition 1 and eliminating part (c) of Lemma 2. A proof that part (b) of Lemma 2 implies part (a) can easily be constructed in a manner similar to the one showing part (a) implies part (b). The main result can then be achieved by eliminating part (c) of Theorem 5 and restating it as follows.

Theorem 5. (altered form) In a ring R, the following are equivalent:
(a) R has the left cancellation property.
(b) R contains no left zero divisors.
(c) R contains no right zero divisors.
(d) R has the right cancellation property.

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