Proving Routh's Theorem Using a Different Approach

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Abstract

The Routh's theorem which has important applications in Fluid mechanics already exists in literature e.g. A.S. Ramsey [1], L.M. Milne – Thomson [2], C.C. Lin [3], and P.G. Saffman [4]. In this paper, Routh's theorem has been proved using a different approach.

ROUTH'S THEOREM

Routh's Stream Function

Let $\psi_1(\xi_1, \eta_1)$ be the stream function of a vortex of strength Γ_1 at the point $Q_1(\zeta_1)$ in the ζ–plane and $\psi_0(\xi_1, \eta_1)$ be the stream function of the flow field (i.e. a uniform stream or any other vortex), then the stream function of the combined flow called the Routh's stream function Ψ' is defined as: $\Psi' = \psi_0(\xi_1, \eta_1) + \dfrac{\Gamma_1}{2}\psi_1(\xi_1, \eta_1)$

STATEMENT

Under a conformal transformation $z = f(\zeta)$, which gives motion in the z – plane from that in the ζ – plane, the Routh function for the new motion is given by

$$\Psi = \Psi' + \frac{\Gamma_1^2}{4\pi}\log\left|\frac{d\zeta_1}{dz_1}\right|$$

DERIVATION

Let there be a vortex of strengths Γ_1 placed at the point $P_1(z_1)$ in the z – plane . Then the complex velocity potential $w(z)$ in the z – plane at the point (x, y) is given by $w(z) = w_1(z) + \dfrac{i\Gamma_1}{2\pi}\log(z - z_1)$ (1)

where $w_1(z)$ is the complex velocity potential of an other flow field (i.e. a uniform stream or any other vortex).

Let there be corresponding vortex of strength Γ_1 placed at the point $Q_1(\zeta_1)$ in the ζ – plane under the conformal transformation $\zeta = f(z)$.

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Then the complex velocity potential w (ζ) in the ζ – plane at point (ξ , η) is given by: $w(\zeta) = w_1(\zeta) + \frac{i\,\Gamma_1}{2\,\pi}\log(\zeta - \zeta_1)$ (2)

Now taking the conjugate on both sides of equation (1), we get

$$w(z) = \bar{w}_1\left(\bar{z}\right) - \frac{i\,\Gamma_1}{2\,\pi}\log\left(\overline{z - z_1}\right) \qquad (3)$$

Subtracting equation (3) from equation (1), we have

$$w(z) - \bar{w}\left(\bar{z}\right) = w_1(z) - \bar{w}_1\left(\bar{z}\right) + i\,\frac{\Gamma_1}{2\,\pi}\log(z - z_1) + i\,\frac{\Gamma_1}{2\,\pi}\log\left(\overline{z - z_1}\right)$$

$$2\,i\,\psi(x, y) = 2\,i\,\psi_1(x, y) + \frac{i\,\Gamma_1}{2\,\pi}\log(z - z_1)\left(\overline{z - z_1}\right)$$

or $\quad 2\,i\,\psi(x, y) = 2\,i\,\psi_1(x, y) + \frac{i\,\Gamma_1}{2\,\pi}\log|z - z_1|^2$

or $\quad 2\,i\,\psi(x, y) = 2\,i\,\psi_1(x, y) + \frac{i\,\Gamma_1}{\pi}\log|z - z_1|$

or $\quad \psi(x, y) = \psi_1(x, y) + \frac{\Gamma_1}{2\,\pi}\log|z - z_1| \qquad (4)$

Similarly, $\psi(\xi, \eta) = \psi_1(\xi, \eta) + \frac{\Gamma_1}{2\,\pi}\log|\zeta - \zeta_1| \qquad (5)$

Since the complex velocity potentials w are equal at the corresponding points in the two planes, it follows from equation (1) and (2), that w (z) = w (ζ)

therefore $\quad \psi(x, y) = \psi(\xi, \eta) \qquad (6)$

From equations (4), (5), and (6), we get

$$\psi_1(x, y) + \frac{\Gamma_1}{2\,\pi}\log|z - z_1| = \psi_1(\xi, \eta) + \frac{\Gamma_1}{2\,\pi}\log|\zeta - \zeta_1|$$

$$\psi_1(x, y) = \psi_1(\xi, \eta) + \frac{\Gamma_1}{2\,\pi}\log|\zeta - \zeta_1| - \frac{\Gamma_1}{2\,\pi}\log|z - z_1|$$

$$= \psi_1(\xi, \eta) + \frac{\Gamma_1}{2\,\pi}\log\frac{|\zeta - \zeta_1|}{|z - z_1|}$$

or $\quad \psi_1(x, y) = \psi_1(\xi, \eta) + \frac{\Gamma_1}{2\,\pi}\log\left|\frac{\zeta - \zeta_1}{z - z_1}\right| \qquad (7)$

Taking limit as $z \to z_1$, $\zeta \to \zeta_1$, we have

$$\psi_1(x_1, y_1) = \psi_1(\xi_1, \eta_1)$$

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$$+ \lim_{\substack{z \to z_1 \\ \zeta \to \zeta_1}} \frac{\Gamma_1}{2\pi} \log \left| \frac{\zeta - \zeta_1}{z - z_1} \right|$$

$$= \psi_1 (\xi_1, \eta_1) + \frac{\Gamma_1}{2\pi} \log \left| \lim_{\substack{z \to z_1 \\ \zeta \to \zeta_1}} \left(\frac{\zeta - \zeta_1}{z - z_1} \right) \right| \qquad (8)$$

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Since $f(z) = f(z_1 + z - z_1) = f[z_1 + (z - z_1)]$

or $f(z) = f(z_1) + \dfrac{(z - z_1)}{1!} f'(z_1) + \dfrac{(z - z_1)^2}{2!} f''(z_1) + \ldots\ldots$ (9)

But $f(z) = \zeta$ and $f(z_1) = \zeta_1$

Also $f'(z_1) = \left(\dfrac{d\zeta}{dz} \right)_{z = z_1}$, $f''(z_1) = \left(\dfrac{d^2\zeta}{dz^2} \right)_{z = z_1}$, ……

Thus from equation (9) , we get

$$\zeta = \zeta_1 + (z - z_1) \left(\frac{d\zeta}{dz} \right)_{z = z_1} + \frac{1}{2} (z - z_1)^2 \left(\frac{d^2\zeta}{dz^2} \right)_{z = z_1} + \ldots\ldots$$

or $$\zeta - \zeta_1 = (z - z_1) \left(\frac{d\zeta}{dz} \right)_{z = z_1} + \frac{1}{2} (z - z_1)^2 \left(\frac{d^2\zeta}{dz^2} \right)_{z = z_1} + \ldots\ldots$$

Dividing both sides by $(z - z_1)$, we get

$$\frac{\zeta - \zeta_1}{z - z_1} = \left(\frac{d\zeta}{dz} \right)_{z = z_1} + \frac{1}{2} (z - z_1) \left(\frac{d^2\zeta}{dz^2} \right)_{z = z_1} + \ldots\ldots$$

Now taking limit as $z \to z_1$, $\zeta \to \zeta_1$, we have

$$\lim_{\substack{z \to z_1 \\ \zeta \to \zeta_1}} \left(\frac{\zeta - \zeta_1}{z - z_1} \right) = \lim_{\substack{z \to z_1 \\ \zeta \to \zeta_1}} \left[\frac{d\zeta}{dz} + \frac{1}{2} (z - z_1) \frac{d^2\zeta}{dz^2} + \ldots\ldots \right]$$

$$= \left(\frac{d\zeta}{dz} \right)_{\substack{z = z_1 \\ \zeta = \zeta_1}} = \frac{d\zeta_1}{dz_1} \text{ or } \left| \lim_{\substack{z \to z_1 \\ \zeta \to \zeta_1}} \left(\frac{\zeta - \zeta_1}{z - z_1} \right) \right| = \left| \frac{d\zeta_1}{dz_1} \right| \qquad (10)$$

From equations (8) and (10), we get

$$\psi_1 (x_1, y_1) = \psi_1 (\xi_1, \eta_1) + \frac{\Gamma_1}{2\pi} \log \left| \frac{d\zeta_1}{dz_1} \right| \quad (11)$$

Multiplying both sides of equation (11) by $\frac{\Gamma_1}{2}$, we have

$$\frac{\Gamma_1}{2} \psi_1 (x_1, y_1) = \frac{\Gamma_1}{2} \psi_1 (\xi_1, \eta_1) + \frac{\Gamma_1^2}{4\pi} \log \left| \frac{d\zeta_1}{dz_1} \right| \quad (12)$$

Adding $\psi_0 (x_1, y_1)$ on both sides of equation (12), we have

$$\psi_0 (x_1, y_1) + \frac{\Gamma_1}{2} \psi_1 (x_1, y_1) = \psi_0 (x_1, y_1) + \frac{\Gamma_1}{2} \psi_1 (\xi_1, \eta_1) + \frac{\Gamma_1^2}{4\pi} \log \left| \frac{d\zeta_1}{dz_1} \right| \quad (13)$$

But $\psi_0 (x_1, y_1) = \psi_0 (\xi_1, \eta_1)$

Then equation (13) becomes

$$\psi_0 (x_1, y_1) + \frac{\Gamma_1}{2} \psi_1 (x_1, y_1) = \psi_0 (\xi_1, \eta_1) + \frac{\Gamma_1}{2} \psi_1 (\xi_1, \eta_1) + \frac{\Gamma_1^2}{4\pi} \log \left| \frac{d\zeta_1}{dz_1} \right| \quad (14)$$

Since $\Psi = \psi_0 (x_1, y_1) + \frac{\Gamma_1}{2} \psi_1 (x_1, y_1)$ (15)

Similarly $\Psi' = \psi_0 (\xi_1, \eta_1) + \frac{\Gamma_1}{2} \psi_1 (\xi_1, \eta_1)$ (16)

From equations (14), (15), and (16) , we get $\Psi = \Psi' + \frac{\Gamma_1^2}{4\pi} \log \left| \frac{d\zeta_1}{dz_1} \right|$

(17)which is the Routh's theorem .

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GENERALIZATION

Let there be n vortices of strengths $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ placed at the points $P_1(z_1)$, $P_2(z_2) \ldots, P_n(z_n)$ respectively in the z – plane .

Let there be corresponding vortices of strengths $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ placed at the points $Q_1(\zeta_1)$, $Q_2(\zeta_2) \ldots, Q_n(\zeta_n)$ in the ζ – plane under the conformal transformation $\zeta = f(z)$.

Then the generalized form of the Routh's theorem is

$$\Psi = \Psi' + \sum_{i=1}^{n} \frac{\Gamma_i^2}{4\pi} \log \left| \frac{d\zeta_i}{dz_i} \right| \qquad (18)$$

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References

[1] A.S. Ramsey, M.A., "A Treatise on Hydrodynamics", G. Bell and Sons, Ltd. 1942.

[2] L.M. Milne – Thomson, C.B.E. "Theoretical Hydrodynamics", Robert Maclehose and Co. Ltd. The University Press, Glasgow, 1974.

[3] C.C. Lin, "On the Motion of Vortices in two Dimensions – II Some Further Investigations on the Kirchhoff – Routh Function", PNAS, U.S.A., Vol. 27, 1941.

[4] P.G. Saffman, "Vortex Dynamics", Cambridge Monographs on Mechanics and Applied Mathematics, Cambridge University Press, Second Edition, 1992.