On Writing in Mathematics Courses: Closing the Circuit of Understanding Mathematics by the Written Word

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Abstract

This article discusses the importance of writing in full sentences in mathematics courses. The aim is to promote early introduction of level-appropriate rigorous writing in mathematics courses. It may be true that the proposed writing requirements in early courses are not as detailed / intensive as in more advanced courses like Analysis and Abstract Algebra. However, early acquisition of such facility is amenable to a better understanding of and less rote memorization in early concepts. Consequently, it also serves the advanced courses because it gradually builds up to the rigor and sophistication required by later courses. They have learned how to swim and not sink! Just as importantly, this early insistence in full sentence construction and communication should promote the efficacy of mathematics service courses.

Introduction

Human beings are social creatures. As a consequence, they need tools for the communication of their thoughts. There are many kinetic and verbal means of communication. As a means of verbal communication, the written word is most effective in assuring mutual understanding and the sharing of thoughts. In the modern society, we come to know how others think by reading their written work, by listening to them speak, and asking them questions. “Verbal (oral) interactions are often informative, but they are not necessarily a ‘clean’ look at the thoughts of others because our own thoughts are part of the process.” [3, p 32] This mingling of our thoughts with thoughts of the initiator of an oral discussion introduces a certain degree of approximation to the purity of our understanding of the original thoughts. Such an approximation would be good enough for many purposes. However, as educators, and more specifically as mathematicians, “if we really want to know whether another mathematician has proved a theorem, say, we want to see the proof in written form with all the details–and no intervention by others, including ourselves.” [3, p 32, our italics]

Our educational system requires all students to take some mathematics courses. The main purpose is to help students develop their own comprehension and understanding of concepts and logical reasoning. We, as members of a profession, require the written word as the proof of understanding. We would be betraying our profession, and our students, when we do not hold them to the same rigorous criteria as we do ourselves. David Smith says "Why should it be any different with students? If we are to take seriously our goal of having them understand concepts, we need a window on their minds." [3, p 32] We can attain the referred window to students' minds by demanding that they also communicate their thoughts in writing. Since the basic building block for thinking and communication of thoughts is a sentence, we can begin by asking our students to write in complete sentences.

Mutual Understanding Through Writing

Throughout the history of civilization, human beings have sought to create bonds of common understanding by writing things down. Misunderstandings are a normal part of human discourse, and so are their resolutions by the written word. This phenomenon occurs frequently in mathematics; however, examples also abound in business, the legal system, politics, etc. A partially fictitious but quite relevant example from a common business transaction follows.

> A prospective home owner talks to a builder/contractor about constructing a house. After all the verbal communication, both agree that they have the same understanding about what the house will look like and its cost of $500,000. In order to make sure that their thoughts coincide, the home owner writes down a contract which reveals a disparity in their thinking that was not revealed in their oral agreement. They don't share the same idea of what a bathtub is. The owner writes that the bathtub is a 1,000 gallon model. The builder had a somewhat less extravagant model in mind. The builder also says the floor won't support such a tub unless additional support is provided costing an extra $100,000. They negotiate and rewrite the following sentence in the contract: "The bathtub shall have a capacity of 150 gallons, which adds $15,000 to the cost for flooring reinforcement." The circuit of understanding is now closed. Their idea of the bathtub is affirmed through the written word.

As social creatures, humans tend to be agreeable. They too quickly jump to an agreement of a mutual understanding of concepts based on oral expressions of thoughts. A way to remedy this situation is to insist on writing proper and complete sentences. A sentence may be a natural language (English, Spanish, etc.) sentence, an equation, a symbolic logic statement, an algorithmic statement, a statement in a computer program, etc.

The quest for common understanding is an essential part of mathematics. Therefore, it is worthwhile to reexamine the fundamental role of writing in mathematics and its modes of employment.

Writing and Mathematics

A typical experience is seeing students follow the lecture and ensuing discussions; they nod and agree that they have understood the concepts and processes under discussion. However, most complain that when they are to solve a problem on their own, they realize their lack of complete understanding of the related concepts. Herbert Wilf, on receiving his MAA's 1995 distinguished teaching award, says the following about his junior level Analysis class:

> I wish I had a nickel for every student who has told me, "I understand it. I just can't really say it!" That's a very human feeling. The problem is that unless you can say this thing, you won't even be able to understand the next thing. [5, p 24]

The ability to "say it", in writing, is a general problem in mathematics education. This is one of the concerns that have prompted recent educational reform efforts: 'Writing Across the Curriculum', 'Calculus Reform', etc. Herbert Wilf calls for reforming the Analysis course by starting the course with writing exercises, "writing out mathematical sentences and paragraphs, in full, in which the mathematics in those sentences will be very familiar to them." [5, p 24] We take three points to emphasize from this suggestion: a) students should be required to write in full sentences, b) writing in complete sentences should begin as soon as possible, and c) it is never too late to start writing to express mathematics. If (b) is achieved, in time it will render (c) obsolete.

Let's look at an early stage in everyone's mathematical development, whether they be a mathematician or not.

Early Mathematical Sentences

The source of the problem lies in early education of a child when the child learns basic addition and multiplication (the tables). Take the following addition and multiplication sentences:

$$2+5=7,\ \text{"two plus five equals seven"}$$

$$2\times 5=10,\ \text{"two times five equals ten"}$$

For the purpose of rote memorization, the sentence

$$2+5=7,\ \text{"two plus five equals seven"}$$

becomes "two plus five seven" in order to save steps and time. In doing so the verb of the sentence has been lost. Even worse is when we venture to multiplication. The sentence $2\times 5=10$, "two times five equals ten", becomes "two five ten". Now the whole thought process is reduced to a memorization of a triplet of juxtaposed numbers.

This shorthand approach is appropriate for the purpose of memorizing the table(s), but it is not mathematics, the science of thinking and problem solving. Many of our students never make the transition back to associating mathematics with a language of thought, which inherently has full sentence structure.

Teachers of mathematics are under pressure to cover the topical objectives of mandated curricula. However, we must be careful not to constantly sacrifice the natural and fundamental sentence structure of mathematics in order to achieve short-term objectives. Removal of verbs and other action words / phrases leads to juxtaposition of numbers, symbols, steps in the process of an algorithm, etc., compromises the integrity of the sentence structure and undermines the thought process.

Often students seem to be looking for a chain of operations in what to do next. We, on the other hand, first look for understanding of a concept which dictates the chain of operations. Of course, the lack of proper expressions of thoughts through the use of full sentences is a major contributor to this phenomenon. (Unfortunately, we unintentionally contribute to this process because of exigencies of mass education.)

Mathematics teachers are horrified by seeing a student writing the following sequence of steps.

$$
\begin{array}{lllll}
5x & +3 & -2x & = & 5 \\
+2x & & +2x & & \\
\hline
7x & +3 & & = & 5 \\
 & -3 & & & -3 \\
\hline
7x & & & = & 2 \\
\hline
\div 7 & & & & \div 7 \\
\hline
x & & & = & \frac{2}{7}
\end{array}
$$

What happened here? The student never recognized the equal sign "=" as the verb of the sentence. The student's subsequent efforts to isolate x violated this fundamental concept and definition of the statement of the problem. Thus, it is missing the premise of the problem that dooms the student to logical failure.

These examples from arithmetic to algebra of omitting verbs from very early in the educational life of a student show the compounding effect over the years of violating the integrity of the sentence structure. Lack of sentence structure (verbs) means lack of thoughts and consequently lack of development of mathematical thought processes.

Crucial Role of the Verb

Complete sentences have verbs.
Ask a student to read

$$6 < n.$$

The response is “six less than n”, which actually stands for the expression $n - 6$. Whereas, $6 < n$ is actually a complete sentence and you most probably meant “Six is less than n.”, with the verb “is” (actually the verb phrase “is less than”) being the essence that completes the sentence and hence facilitates understanding of the thoughts being communicated. This problem is more pronounced when the problem size is compounded.

Let’s take the expression

$$n - (n - 6)$$

that the author of a text has in mind as part of the setup for a problem. The student is asked to translate this idea from the verbal expression “a number minus six less than the number” back into a symbolic expression. Several students respond (perhaps correctly!) by

$$n - 6 < n$$

Again, the point missed is that the students relate the phrase “less than” to the symbol “<”, which in many textbooks is often referred to as “less than”. Many of us casually use the same connotation for the symbol in our instructions and oral communications. The students with such misconceptions are blamed unjustly for the mistake, whereas it may be our fault and that of our textbooks for creating the ambiguous situation.

If students do not develop, from early on, the awareness to distinguish between mathematical sentences and non-sentence expressions, it can have serious consequences later when they study or use more advanced mathematics.

For example, in set theory, “$\subseteq$” is a verb similar to “=”, whereas “$\cup$”and “$\cap$” are binary operators like “+”. Thus, when A and B are sets, “$A \subseteq B$” is a sentence (a statement), whereas $A \cup B$ and $A \cap B$ are expressions representing sets. Yet, even in courses like Discrete Mathematics, Abstract Algebra, or Intermediate Analysis (Advanced Calculus) where students are required to write paragraph proofs using set theoretic language, students sometimes write phrases like “suppose $A \cap B$,” “since $A \cap B$,” or “if $A \cap B$.” Also, misuse of logical connective “$\Rightarrow$” like “$5(x+2) \Rightarrow 5x+10$” is not uncommon in courses like College Algebra or Calculus, where students use “⇒” instead of “=”. These students are treating expressions like “A∩B”, “5(x+2)”, or “5x+10” as if they were sentences.

Conversely, in number theory, the symbol “|” is a verb / verb phrase that means “divides” or “is a divisor of”. Namely, when a and b are integers, “a|b” is a sentence that says “a divides b(a is a divisor of b).” Yet, some students frequently write equalities like “3|6=2.” Those students are mistaking “|” for a binary operator, and as a consequence, treating a sentence “3|6” as if it were an expression that evaluates to a number. Similarly, some students may solve equations as in “3x = 6 = x = 2”, where the middle “=” should be replaced by either the implication “⇒” or the “therefore” symbol “∴”. Here, sentences “3x=6”and “x=2” are being treated as if they were (non-sentence) expressions.

Of course, one aspect of mistakes like these is misunderstanding of the meaning of symbols, e.g. ⊆,⇒, =, or |. But a more fundamental underlying cause is that these students have not developed the sense to distinguish verbs from operators, or sentences from non-sentences. If they had, misuses like the examples above would be largely avoided, even if the precise meaning of each symbol were to be misunderstood. If students practice from the early grades to write in full sentences and thus are made aware that verbs like “=”or “<`” have sentence-forming power whereas operators like “+” or “×” do not, then they are much less likely to commit a fundamental error of treating non-sentences as sentences, or vice versa.

Suggestions to Remedy the Situation

Students of all levels should be required to write mathematics in complete sentences. It is never too early to start, nor is it ever too late to include full sentence writing exercises in mathematics courses. This does not require extensive or expensive changes to the curriculum. It does mean requiring some sentence and paragraph writing in every mathematics course. The earlier students begin to write in sentences and then paragraphs, the less obtrusive the requirement will be in later courses, in some of which writing is indispensable, [5, p 24].

If we expect our students to write in sentences, it is obvious that we, the teachers, should model writing mathematics in complete sentences – that means on the board, too. But as importantly, we need to carefully read and constructively comment on student writing. Writing may be considered by some to be a passive activity, but it is essential to understanding, inclusive of understanding oneself. The following excerpt from Rubik, the inventor of Rubik’s Magic Cube, puts it elegantly.

> I am an active person. Activity is my element. Writing is an extremely passive form of action for me. But how deferent and

> how wonderful it is to bring something into existence, to give form to a substance, to make it purposeful and attractive to the eye and the hand. So why did I, in spite of everything, decide to write about it myself? I must admit my motives were rather selfish – I wanted to understand myself. After all, the easiest way to understand something yourself is to try and explain it to somebody else. [2, p 1]

The implication of requiring writing in early mathematics courses is multifaceted.

- By writing in early mathematics courses, students become active learners, because students can only then have the opportunity to clearly see and be sure of their own thinking.
- By careful reading of students' written work, we have a clear window to observe their thoughts. David Smith suggests writing as a window to the mind.

> They have to tell us what and how they are thinking. We can listen to them talk, and we can ask them questions. That is a start, but our own thoughts are inevitably part of that process. The cleanest window we have is the student writing. [3, p 32]

- Through constructive commentary on their writing, students will have a greater opportunity to evolve dynamically from their current state of mathematical comprehension and thinking.
- Students engaged in this process become practicing mathematicians, whatever their level may be. This fosters the role of mathematics as service courses, core or otherwise, say, for arts, business, engineering, languages, medical sciences, sciences, etc. Early mathematics practitioners obviously benefit the mathematics community. Also, they enhance and enlarge the pool whence we can draw mathematics majors and future professional mathematicians.
- Reading of written work in mathematics is potentially the most labor-intensive activity among all fields of intellectual pursuit. This is a one-on-one process, because in mathematics we require continuity of thoughts both global and local, both macro and micro. A small flaw anywhere in the process may negate the whole effort.

As a consequence of the above recommendations, especially the latter, we need to reexamine two aspects of our mass education's typical classroom constraints: class size and course content. In both cases, we must emphasize quality over quantity. If you sacrifice quality in favor of quantity, you merely achieve transient results. True understanding has not occurred. Our resolve as educators should be that our students retain our teachings longer. “I shall light a candle of understanding in thin heart, which shall not be put out”, Apocrypha, II Esdras, XIV, 25.

- It is already known that a common characteristic of effective undergraduate mathematics programs is small class size. Most post calculus courses intrinsically require a moderate to extensive amount of writing, however, in recent memory such requirements have been mostly absent from calculus and earlier courses. It is with this view point that the findings from the case studies in effective mathematics programs says, “The class sizes are normally small to moderate. Multiple sections of post calculus courses are scheduled if the enrollment is over 30.” [4, p 23] In order to implement ‘full sentence’ writing assignments throughout the mathematics curriculum, students must associate and become fluent in writing meaningful mathematics as early as possible.

  Analogously, English classes that focus on the teaching of writing appear early in a student’s educational life. Enrollments for these writing courses are generally mandated to be small. At the authors’ institutions, these class sizes are approximately 20. The later English courses do have major writing requirements, but by this time the students know how to write, and as importantly associate writing with their discipline.
  Early introduction and requirement of writing in mathematics courses must be afforded the same nurturing environment as with small class enrollment and individual attention. Mathematics classes that focus on the teaching of writing must appear early in a student’s mathematical education. Enrollment for these mathematical writing courses should be mandated to be small, a class size of 15—25 is essential. The teaching faculty must respond to a student’s written efforts with the same intensity and concentration as they would a practicing mathematician’s work. Once students gain fluency in mathematical writing, and associate writing and mathematics, they can reap the benefit of their education in their future courses and walks of life. Students will enter subsequent mathematics courses as experienced writers and thinkers of quality mathematics.

- We may be able to sacrifice breadth of mathematical coverage to gain a greater depth in understanding. But, we cannot sacrifice understanding to gain greater coverage of mathematical topics, because the integrity of the whole foundation of mathematical thoughts, reasoning and understanding is violated. The number and kind of topical objectives in such a course or courses must be carefully considered. They must be appropriate to and consistent with the mathematical ideals of our educational system. In the end, we may sacrifice fewer students!

Conclusion

Writing is a powerful tool. A master writer forces the disciplined reader to have the thoughts the author intended to convey. As mathematicians we enjoy reading and sharing the exact thoughts of a Newton or Gauss. Why should our students, at any mathematical level, be deprived of giving or receiving this pleasure? Mathematics can be an enjoyable experience. Historically, games and puzzles have been everyday entertainment and recreational activities. Currently, our typical mathematics courses are enjoyable neither for students nor for teachers. We should help restore mathematical confidence in people so that they can enjoy mathematics again.

Learning is fun. If we get students to practice mathematics in the same way as a professional mathematician does, they will be writing, understanding, learning and communicating their thoughts. This is an addictive, rewarding, and enjoyable activity. There is no reason a social and logical being should be unhappy to learn. Recall how ecstatic and enthusiastic Archimedes was when he shouted "Eureka! Eureka!" running naked from the public bath to write down his discovery. He had learned something that day and probably he wanted to make sure he understood it himself.

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