Cyclic Product Groups

Richard Alan Winton, Ph.D. †

Abstract

Necessary and sufficient conditions for a product of groups to be cyclic appear in various forms throughout the literature. Furthermore, the conditions presented vary greatly from quite simple to very general. However, the factor groups are routinely assumed to have cyclic properties which are not required for necessary conditions for the product to be cyclic. In this paper we will establish necessary and sufficient conditions for the product of finite groups to be cyclic, while assuming the factor groups are cyclic only for the sufficient conditions that the product is cyclic.

Introduction

Some texts provide only sufficient conditions for special cases of groups of integers modulo n. For example, if $\gcd\{m,n\} = 1$, then $\mathbf{Z}_m \times \mathbf{Z}_n \cong \mathbf{Z}_{mn}$ [1, p. 198, lemma], and so $\mathbf{Z}_m \times \mathbf{Z}_n$ is cyclic of order mn [11, p. 113, Theorem 1]. This basic result can be generalized in several ways. In some cases the converse is included to read that $\mathbf{Z}_m \times \mathbf{Z}_n$ is cyclic if and only if $\gcd\{m,n\} = 1$ [10, p. 98, no. 13,14]. In other cases, the result is not stated biconditionally, but is extended to a finite number of factor groups with special moduli. Specifically, if $\{p_i\}_{i=1}^{r}$ is a collection of distinct primes and $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, then $\prod_{i=1}^{r} \mathbf{Z}_{p_i^{k_i}} \cong \mathbf{Z}_n$ [1, p. 199, Theorem 3.6], which implies that $\prod_{i=1}^{r} \mathbf{Z}_{p_i^{k_i}}$ is cyclic of order n [11, p. 113, Theorem 1]. However, this statement is not biconditional, and the moduli are required to be prime powers rather than only pairwise relatively prime. Both of these limitations are removed with the more general result that $\prod_{i=1}^{n} \mathbf{Z}_{m_i}$ is cyclic if and only if $\gcd\{m_i, m_j\} = 1$ whenever $1 \le i < j \le n$ [4, p. 80, corollary].

Another direction in which to generalize the result in [1, p. 198, lemma] is to replace the reference to $\mathbf{Z}_m$ and $\mathbf{Z}_n$ with arbitrary finite groups. In this manner, if G and H are finite cyclic groups and $\gcd\{|G|,|H|\} = 1$, then G×H is cyclic [2, p. 231, lemma]. This result is quite limited, providing only sufficient conditions for the product of two finite cyclic groups to be cyclic. Something of a converse appears in [7, p. 178, no. 40], stating that if G×H is cyclic, then G and H are cyclic. These two results can be combined to show that if G and H are finite cyclic groups, then G×H is cyclic if and only if $\gcd\{|G|,|H|\} = 1$ ([5, p.

107, Theorem 8.2],[6, p. 112, no. 2],[8, p. 63, no. 5]). This biconditional result is extended to a finite number of groups by showing that if G_i is a finite cyclic group for $1 \le i \le n$, then $\prod_{i=1}^{n} G_i$ is cyclic if and only if $\gcd\{|G_i|,|G_j|\} = 1$ whenever $1 \le i < j \le n$ ([5, p. 107, corollary],[12, p. 57, Theorem 6.1(ii)]).

In all of the above cases, the finite cyclic nature of the factor groups G_i is assumed for both the necessity and sufficiency that $\prod_{i=1}^{n} G_i$ is cyclic. The main result of this paper assumes only that G_i is a finite group for $1 \le i \le n$ for necessary conditions that $\prod_{i=1}^{n} G_i$ is cyclic. The property that G_i is cyclic for $1 \le i \le n$ is assumed only for sufficient conditions that $\prod_{i=1}^{n} G_i$ is cyclic. First, however, we need a lemma for the inductive proof to follow.

Preliminary Results

Lemma 1: Suppose G and H are finite groups. Then G×H is cyclic if and only if G and H are cyclic and $\gcd\{|G|,|H|\} = 1$.

Proof: If G and H are cyclic and $\gcd\{|G|,|H|\} = 1$, then G×H is cyclic ([2, p. 231, lemma],[5, p. 107, corollary],[12, p. 57, Theorem 6.1(ii)]).

Conversely, if G×H is cyclic, then there is an element $(u,v) \in G\times H$ such that $|(u,v)| = |G\times H| = |G|\cdot|H|$. Then $u\in G$, $v\in H$, and $|(u,v)| = \operatorname{lcm}\{|u|,|v|\}$ ([4, p. 81, Theorem 8.3],[5, p. 106, Theorem 8.1]). Furthermore, $|u|$ and $|v|$ divide $|G|$ and $|H|$, respectively ([3, p. 81, corollary],[1, p. 130, Theorem 2.41]), so that $|G| = |u|\cdot m$ and $|H| = |v|\cdot n$ for some positive integers m and n. Thus $\operatorname{lcm}\{|u|,|v|\} = |(u,v)| = |G|\cdot|H| = (|u|\cdot m)(|v|\cdot n)$,

so that $\dfrac{|u|\cdot|v|}{\gcd\{|u|,|v|\}} = |u|\cdot|v|\cdot m\cdot n$.

Consequently, $m\cdot n\cdot \gcd\{|u|,|v|\} = 1$, and so $m = n = \gcd\{|u|,|v|\} = 1$. Therefore $|G| = |u|$, $|H| = |v|$, and $\gcd\{|G|,|H|\} = 1$. Since G and H are finite, $|G| = |u|$, and $|H| = |v|$, then $G = \langle u\rangle$ and $H = \langle v\rangle$. Hence G and H are cyclic and $\gcd\{|G|,|H|\} = 1$.

Main Result

We are now prepared to present the main result of this paper. Theorem 2 extends the result of Lemma 1 to a finite number of groups.

Theorem 2: Suppose G_i is a finite group for $1 \le i \le n$. Then $\prod_{i=1}^{n} G_i$ is cyclic (of order $\prod_{i=1}^{n} |G_i|$) if and only if G_i is cyclic for $1 \le i \le n$ and $\gcd\{|G_i|, |G_j|\} = 1$ whenever $1 \le i < j \le n$.

Proof: Clearly Theorem 2 is true for n = 1, and is also true for n = 2 according to Lemma 1. Now suppose Theorem 2 is true for any collection of n–1 finite groups.

If G_i is a finite group for $1 \le i \le n$, then $\prod_{i=1}^{n} G_i \cong \prod_{i=1}^{n-1} G_i \times G_n$ by a natural extension of [9, p. 219, Theorem 5.30]. Thus $\prod_{i=1}^{n} G_i$ is cyclic if and only if $\prod_{i=1}^{n-1} G_i \times G_n$ is cyclic if and only if $\prod_{i=1}^{n-1} G_i$ and G_n are cyclic and $\gcd\left\{\left|\prod_{i=1}^{n-1} G_i\right|, |G_n|\right\} = 1$ by Lemma 1. However, $\gcd\left\{\left|\prod_{i=1}^{n-1} G_i\right|, |G_n|\right\} = 1$ if and only if $\gcd\left\{\prod_{i=1}^{n-1} |G_i|, |G_n|\right\} = 1$ if and only if $\gcd\{|G_i|, |G_n|\} = 1$ for $1 \le i \le n-1$. Therefore $\prod_{i=1}^{n} G_i$ is cyclic if and only if $\prod_{i=1}^{n-1} G_i$ and G_n are cyclic and $\gcd\{|G_i|, |G_n|\} = 1$ for $1 \le i \le n-1$.

Furthermore, by the induction hypothesis, $\prod_{i=1}^{n-1} G_i$ is cyclic if and only if G_i is cyclic for $1 \le i \le n-1$ and $\gcd\{|G_i|, |G_j|\} = 1$ whenever $1 \le i < j \le n-1$. Therefore $\prod_{i=1}^{n} G_i$ is cyclic if and only if G_i is cyclic for $1 \le i \le n-1$, G_n is cyclic, $\gcd\{|G_i|, |G_j|\} = 1$ whenever $1 \le i < j \le n-1$, and $\gcd\{|G_i|, |G_n|\} = 1$ for $1 \le i \le n-1$, which is true if and only if G_i is cyclic for $1 \le i \le n$ and $\gcd\{|G_i|, |G_j|\} = 1$ whenever $1 \le i < j \le n$.

The result follows by induction.

Alternate Proof

As mentioned above, it is well known that if G_i is a finite cyclic group for $1 \le i \le n$, then $\prod_{i=1}^{n} G_i$ is cyclic if and only if $\gcd\{|G_i|, |G_j|\} = 1$ whenever $1 \le i < j \le n$. This result provides an alternate approach to Theorem 2. Alternate Proof of Theorem 2: Suppose G_i is a finite group for $1 \le i \le n$.

If G_i is cyclic for $1 \le i \le n$ and $\gcd\{|G_i|, |G_j|\} = 1$ whenever $1 \le i < j \le n$, then $\prod_{i=1}^{n} G_i$ is cyclic ([5, p. 107, corollary],[12, p. 57, Theorem 6.1(ii)]).

Conversely, suppose that $\prod_{i=1}^{n} G_i$ is cyclic and G_i has identity element e_i for $1 \le i \le n$. If $1 \le k \le n$, then $H_k = \{(e_1, \cdots, e_{k-1}, x, e_{k+1}, \cdots, e_n) | x \in G_k\}$ is a subgroup of $\prod_{i=1}^{n} G_i$ and $H_k \cong G_k$ [9, p. 217, Theorem 5.28]. Furthermore, H_k is cyclic since $\prod_{i=1}^{n} G_i$ is cyclic [11, p. 114, Theorem 2]. Therefore G_k is also cyclic since $G_k \cong H_k$.

Thus G_i is cyclic for $1 \le i \le n$. Since G_i is also finite for $1 \le i \le n$ and $\prod_{i=1}^{n} G_i$ is cyclic, then $\gcd\{|G_i|, |G_j|\} = 1$ whenever $1 \le i < j \le n$ ([5, p. 107, corollary],[12, p. 57, Theorem 6.1(ii)]). The result follows.

† *Richard Winton*, *Ph.D.*, Tarleton State University, USA

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