Transformation of Equation Satisfied by Stokes' Stream Function from Cylindrical to Elliptic Coordinates

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Abstract

The equation satisfied by the Stokes' stream function for irrotational motion and its transformation from cylindrical to elliptic coordinate exist in literature. In this paper, a different approach to transform this equation from cylindrical to elliptic coordinates is presented.

Derivation

We know that the equation satisfied by the stream function ψ for irrotational motion in cylinderical coordinates is

$$\frac{\partial^2 \psi}{\partial z^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial r^2} = 0 \qquad (1)$$

It is required to transform this equation in elliptic coordinates. We transform the independent variables (z, r) in equation (1) to elliptic coordinates (ξ, η). Suppose that z1 and ζ are two complex variables defined by

$$z_1 = z + i\,r \quad \text{and} \quad \zeta = \xi + i\,\eta$$

where z, r, ξ, η are real variables. Then we can draw two complex planes, one called the z_1–plane
(or $z\,r$ – plane), the other called the ζ–plane
(or $\xi\,\eta$ – plane). Suppose that ζ is related to z_1 by means of the transformation

$$\zeta = f(z_1) \qquad (2)$$

If $f(z_1)$ is a single valued function of z_1, then to each point in the z_1–plane, there corresponds one and only one point in the ζ–plane. In this way, a curve C (or region R) in the z_1 plane is mapped into a curve C' (or region R') in the ζ–plane and conversely as shown in the figure.

We can write equation (2) as $\xi + i\,\eta = f(z + i\,r) = \xi(z, r) + i\,\eta(z, r)$, which implies

$\xi = \xi(z, r)$ and $\eta = \eta(z, r)$ (3)

Equation (3) are called the transformation equations from the z r – plane to the ξ η – plane.

Since f (z_1) is a single–valued function of z1, we can define the inverse transformation from the

ζ–plane to z_1–plane using equation (2) as

$z_1 = g(\zeta)$ (4)

or $z + i\,r = g(\xi + i\,\eta) = z(\xi, \eta) + i\,r(\xi, \eta)$, which gives

$z = z(\xi, \eta)$ and $r = r(\xi, \eta)$ (5)

From equation (1), we see that $\psi = \psi(z, r) = \psi\left[z(\xi, \eta),\ r(\xi, \eta)\right]$using equation (5)

By chain–rule, we have

$$\frac{\partial \psi}{\partial z} = \frac{\partial \psi}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial z} \quad (6)$$

And $\dfrac{\partial \psi}{\partial r} = \dfrac{\partial \psi}{\partial \xi}\dfrac{\partial \xi}{\partial r} + \dfrac{\partial \psi}{\partial \eta}\dfrac{\partial \eta}{\partial r}$ (7)

Also, $\dfrac{\partial^2 \psi}{\partial z^2} = \dfrac{\partial \psi}{\partial \xi}\dfrac{\partial^2 \xi}{\partial z^2} + \dfrac{\partial \xi}{\partial z}\dfrac{\partial}{\partial z}\left(\dfrac{\partial \psi}{\partial \xi}\right) + \dfrac{\partial \psi}{\partial \eta}\dfrac{\partial^2 \eta}{\partial z^2} + \dfrac{\partial \eta}{\partial z}\dfrac{\partial}{\partial z}\left(\dfrac{\partial \psi}{\partial \eta}\right)$

$$= \frac{\partial \psi}{\partial \xi}\frac{\partial^2 \xi}{\partial z^2} + \frac{\partial \xi}{\partial z} \times \left[\frac{\partial}{\partial \xi}\left(\frac{\partial \psi}{\partial \xi}\right)\frac{\partial \xi}{\partial z} + \frac{\partial}{\partial \eta}\left(\frac{\partial \psi}{\partial \xi}\right)\frac{\partial \eta}{\partial z}\right]$$

$$+ \frac{\partial \psi}{\partial \eta}\frac{\partial^2 \eta}{\partial z^2} + \frac{\partial \eta}{\partial z} \times \left[\frac{\partial}{\partial \xi}\left(\frac{\partial \psi}{\partial \eta}\right)\frac{\partial \xi}{\partial z} + \frac{\partial}{\partial \eta}\left(\frac{\partial \psi}{\partial \eta}\right)\frac{\partial \eta}{\partial z}\right]$$

$$= \frac{\partial \psi}{\partial \xi}\frac{\partial^2 \xi}{\partial z^2} + \frac{\partial \xi}{\partial z}\left[\frac{\partial^2 \psi}{\partial \xi^2}\left(\frac{\partial \xi}{\partial z}\right) + \frac{\partial^2 \psi}{\partial \eta\,\partial \xi}\frac{\partial \eta}{\partial z}\right] + \frac{\partial \psi}{\partial \eta}\frac{\partial^2 \eta}{\partial z^2} + \frac{\partial \eta}{\partial z}$$

$$\left[\frac{\partial^2 \psi}{\partial \xi\,\partial \eta}\frac{\partial \xi}{\partial z} + \frac{\partial^2 \psi}{\partial \eta^2}\frac{\partial \eta}{\partial z}\right] \quad (8)$$

Similarly, $\frac{\partial^2 \psi}{\partial r^2} = \frac{\partial \psi}{\partial \xi}\frac{\partial^2 \xi}{\partial r^2} + \frac{\partial \xi}{\partial r} \times \left[\frac{\partial^2 \psi}{\partial \xi^2}\frac{\partial \xi}{\partial r} + \frac{\partial^2 \psi}{\partial \eta \, \partial \xi}\frac{\partial \eta}{\partial r} \right] +$

$$\frac{\partial \psi}{\partial \eta}\frac{\partial^2 \eta}{\partial r^2} + \frac{\partial \eta}{\partial r}\left[\frac{\partial^2 \psi}{\partial \xi \, \partial \eta}\frac{\partial \xi}{\partial r} + \frac{\partial^2 \psi}{\partial \eta^2}\frac{\partial \eta}{\partial r} \right] \qquad (9)$$

Substituting equations (7), (8), (9), in equation (1), we get

$$\frac{\partial \psi}{\partial \xi}\frac{\partial^2 \xi}{\partial z^2} + \frac{\partial \xi}{\partial z}\left[\frac{\partial^2 \psi}{\partial \xi^2}\frac{\partial \xi}{\partial z} + \frac{\partial^2 \psi}{\partial \eta \, \partial \xi}\frac{\partial \eta}{\partial z} \right] + \frac{\partial \psi}{\partial \eta}\frac{\partial^2 \eta}{\partial z^2} + \frac{\partial \eta}{\partial z}$$

$$\left[\frac{\partial^2 \psi}{\partial \xi \, \partial \eta}\frac{\partial \xi}{\partial z} + \frac{\partial^2 \psi}{\partial \eta^2}\frac{\partial \eta}{\partial z} \right] - \frac{1}{r}\left(\frac{\partial \psi}{\partial \xi}\frac{\partial \xi}{\partial r} + \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial r} \right) + \frac{\partial \psi}{\partial \xi}\frac{\partial^2 \xi}{\partial r^2} + \frac{\partial \xi}{\partial r}$$

$$\left[\frac{\partial^2 \psi}{\partial \xi^2}\frac{\partial \xi}{\partial r} + \frac{\partial^2 \psi}{\partial \eta \, \partial \xi}\frac{\partial \eta}{\partial r} \right] + \frac{\partial \psi}{\partial \eta}\frac{\partial^2 \eta}{\partial r^2} + \frac{\partial \eta}{\partial r}\left[\frac{\partial^2 \psi}{\partial \xi \, \partial \eta}\frac{\partial \xi}{\partial r} + \frac{\partial^2 \psi}{\partial \eta^2}\frac{\partial \eta}{\partial r} \right] = 0$$

or

$$\frac{\partial \psi}{\partial \xi}\left(\frac{\partial^2 \xi}{\partial z^2} + \frac{\partial^2 \xi}{\partial r^2} \right) + \frac{\partial \psi}{\partial \eta}\left(\frac{\partial^2 \eta}{\partial z^2} + \frac{\partial^2 \eta}{\partial r^2} \right) + \frac{\partial^2 \psi}{\partial \xi^2}\left[\left(\frac{\partial \xi}{\partial z} \right)^2 + \left(\frac{\partial \xi}{\partial r} \right)^2 \right]$$

$$+ 2\frac{\partial^2 \psi}{\partial \xi \, \partial \eta}\left[\frac{\partial \xi}{\partial z}\frac{\partial \eta}{\partial z} + \frac{\partial \xi}{\partial r}\frac{\partial \eta}{\partial r} \right] + \frac{\partial^2 \psi}{\partial \eta^2}\left[\left(\frac{\partial \eta}{\partial z} \right)^2 + \left(\frac{\partial \eta}{\partial r} \right)^2 \right]$$

$$-\frac{1}{r}\frac{\partial \psi}{\partial \xi}\frac{\partial \xi}{\partial r} - \frac{1}{r}\frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial r} = 0 \qquad (10)$$

Since $\zeta = f(z_1)$ is analytic, therefore ξ and η are harmonic.

thus $\frac{\partial^2 \xi}{\partial z^2} + \frac{\partial^2 \xi}{\partial r^2} = 0$

and $\frac{\partial^2 \eta}{\partial z^2} + \frac{\partial^2 \eta}{\partial r^2} = 0$

Also, by Cauchy – Riemann equations, we have

$$\frac{\partial \xi}{\partial z} = \frac{\partial \eta}{\partial r}, \quad \frac{\partial \xi}{\partial r} = -\frac{\partial \eta}{\partial z}$$

Therefore,

$$\left(\frac{\partial \xi}{\partial z} \right)^2 + \left(\frac{\partial \xi}{\partial r} \right)^2 = \left(\frac{\partial \eta}{\partial z} \right)^2 + \left(\frac{\partial \eta}{\partial r} \right)^2 = \left(\frac{\partial \xi}{\partial z} \right)^2 + \left(\frac{\partial \eta}{\partial z} \right)^2 =$$

$$\left| \frac{\partial \xi}{\partial z} + i\frac{\partial \eta}{\partial z} \right|^2 = \left| \frac{\partial}{\partial z} + (\xi + i\,\eta) \right|^2 = \left| \frac{\partial \zeta}{\partial z} \right|^2$$

where $\zeta = \xi + i\,\eta$

$$\text{Also, } \frac{\partial \xi}{\partial z}\frac{\partial \eta}{\partial z} + \frac{\partial \xi}{\partial r}\frac{\partial \eta}{\partial r} = \frac{\partial \xi}{\partial z}\left(-\frac{\partial \xi}{\partial r}\right) + \frac{\partial \xi}{\partial r}\left(\frac{\partial \xi}{\partial z}\right) = -\frac{\partial^2 \xi}{\partial z\,\partial r} + \frac{\partial^2 \xi}{\partial r\,\partial z}$$

$$= -\frac{\partial^2 \xi}{\partial z\,\partial r} + \frac{\partial^2 \xi}{\partial z\,\partial r} = 0$$

Because both ξ, η and their second order partial derivatives are continuous. Thus equation (10) reduces to

$$\left(\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2}\right)\left|\frac{\partial \zeta}{\partial z}\right|^2 - \frac{1}{r}\frac{\partial \psi}{\partial \xi}\frac{\partial \xi}{\partial r} - \frac{1}{r}\frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial r} = 0 \qquad (11)$$

Since $\frac{\partial \xi}{\partial r}\frac{\partial r}{\partial \xi} = 1$ and $\frac{\partial \eta}{\partial r}\frac{\partial r}{\partial \eta} = 1$

Hence equation (11) becomes

$$\left(\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2}\right)\left|\frac{\partial \zeta}{\partial z}\right|^2 - \frac{1}{r}\frac{\partial \psi}{\partial \xi}\frac{\partial \xi}{\partial r}\left(\frac{\partial \xi}{\partial r}\frac{\partial r}{\partial \xi}\right) - \frac{1}{r}\frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial r}\left(\frac{\partial \eta}{\partial r}\frac{\partial r}{\partial \eta}\right) = 0$$

$$\text{or } \left(\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2}\right)\left|\frac{\partial \zeta}{\partial z}\right|^2 - \frac{1}{r}\frac{\partial \psi}{\partial \xi}\left(\frac{\partial \xi}{\partial r}\right)^2\frac{\partial r}{\partial \xi} - \frac{1}{r}\frac{\partial \psi}{\partial \eta}\left(\frac{\partial \eta}{\partial r}\right)^2\frac{\partial r}{\partial \eta} = 0$$

$$\text{or} \left(\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2}\right)\left|\frac{\partial \zeta}{\partial z}\right|^2 - \frac{1}{r}\frac{\partial \psi}{\partial \xi}\left[\left(\frac{\partial \xi}{\partial r}\right)^2 + \left(\frac{\partial \xi}{\partial z}\right)^2 - \left(\frac{\partial \xi}{\partial z}\right)^2\right]\frac{\partial r}{\partial \xi}$$

$$-\frac{1}{r}\frac{\partial \psi}{\partial \eta}\left[\left(\frac{\partial \eta}{\partial r}\right)^2 + \left(\frac{\partial \eta}{\partial z}\right)^2 - \left(\frac{\partial \eta}{\partial z}\right)^2\right]\frac{\partial r}{\partial \eta} = 0$$

$$\text{or} \left(\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2}\right)\left|\frac{\partial \zeta}{\partial z}\right|^2 - \frac{1}{r}\frac{\partial \psi}{\partial \xi}\left[\left|\frac{\partial \zeta}{\partial z}\right|^2 - \left(\frac{\partial \xi}{\partial z}\right)^2\right]\frac{\partial r}{\partial \xi} -$$

$$\frac{1}{r}\frac{\partial \psi}{\partial \eta}\left[\left|\frac{\partial \zeta}{\partial z}\right|^2 - \left(\frac{\partial \eta}{\partial z}\right)^2\right]\frac{\partial r}{\partial \eta} = 0$$

$$\text{or} \left(\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2} - \frac{1}{r}\frac{\partial \psi}{\partial \xi}\frac{\partial r}{\partial \xi} - \frac{1}{r}\frac{\partial \psi}{\partial \eta}\frac{\partial r}{\partial \eta}\right) \times \left|\frac{\partial \zeta}{\partial z}\right|^2 +$$

$$\frac{1}{r}\left(\frac{\partial \psi}{\partial \xi}\frac{\partial \xi}{\partial z}\right)\left(\frac{\partial \xi}{\partial z}\frac{\partial r}{\partial \xi}\right) + \frac{1}{r}\left(\frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial z}\right)\left(\frac{\partial \eta}{\partial z}\frac{\partial r}{\partial \eta}\right) = 0$$

$$\text{or}\left(\frac{\partial^2\psi}{\partial\xi^2}+\frac{\partial^2\psi}{\partial\eta^2}-\frac{1}{r}\frac{\partial\psi}{\partial\xi}\frac{\partial r}{\partial\xi}-\frac{1}{r}\frac{\partial\psi}{\partial\eta}\frac{\partial r}{\partial\eta}\right)\times\left|\frac{\partial\zeta}{\partial z}\right|^2+$$

$$\frac{1}{r}\frac{\partial\psi}{\partial z}\frac{\partial r}{\partial z}+\frac{1}{r}\frac{\partial\psi}{\partial z}\frac{\partial r}{\partial z}=0 \qquad (12)$$

Since z and r are independent variables, therefore $\frac{\partial r}{\partial z}=0$. Equation (12) reduces to

$$\left(\frac{\partial^2\psi}{\partial\xi^2}+\frac{\partial^2\psi}{\partial\eta^2}-\frac{1}{r}\frac{\partial\psi}{\partial\xi}\frac{\partial r}{\partial\xi}-\frac{1}{r}\frac{\partial\psi}{\partial\eta}\frac{\partial r}{\partial\eta}\right)\times\left|\frac{\partial\zeta}{\partial z}\right|^2=0$$

Since $\left|\frac{\partial\zeta}{\partial z}\right|\neq 0$, therefore, $\frac{\partial^2\psi}{\partial\xi^2}+\frac{\partial^2\psi}{\partial\eta^2}-\frac{1}{r}\frac{\partial\psi}{\partial\xi}\frac{\partial r}{\partial\xi}-\frac{1}{r}\frac{\partial\psi}{\partial\eta}\frac{\partial r}{\partial\eta}=0$ (13)

Multiplying both sides of equation (13) by $\frac{1}{r}$, we get

$$\frac{1}{r}\frac{\partial^2\psi}{\partial\xi^2}-\frac{1}{r^2}\frac{\partial\psi}{\partial\xi}\frac{\partial r}{\partial\xi}+\frac{1}{r}\frac{\partial^2\psi}{\partial\eta^2}-\frac{1}{r^2}\frac{\partial\psi}{\partial\eta}\frac{\partial r}{\partial\eta}=0$$

$$\text{or}\ \frac{\partial}{\partial\xi}\left(\frac{1}{r}\frac{\partial\psi}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(\frac{1}{r}\frac{\partial\psi}{\partial\eta}\right)=0 \qquad (14)$$

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