Binet Formulas for Recursive Integer Sequences

Homer W. Austin†
Jathan W. Austin‡

Abstract

Many integer sequences are recursive sequences and can be defined either recursively or explicitly by use of Binet-type formulas. Explorations with Binet's formula can lead to new and interesting recursive sequences. Among these sequences are the side and diagonal numbers of a square. Many relationships exist between these two sequences in the same way that numerous relationships exist between the Fibonacci and Lucas sequences. This paper provides a technique for generating many Binet formulas and thus creating many formulas that can be proved by mathematical induction to generate their respective recursive sequences. The mathematics in this paper can be described as educational mathematics and as such, occupies a crucial place in undergraduate mathematics education in that the methods employed here foster student gains in logical reasoning, inductive and deductive reasoning, and understanding of mathematical induction. The author has successfully used the mathematical content of this paper in classes in discrete mathematics taken by first-year students in college.

Introduction

Integer sequences have been studied in number theory for hundreds of years. Two well-known sequences are the Fibonacci and Lucas sequences. Both can be defined recursively and can be defined explicitly using Binet's formulas. These sequences are special cases of generalized Fibonacci sequences. Less well known are the integer sequences of the side and diagonal numbers of the square. These are special cases of generalized Fibonacci sequences and can also be defined recursively or with Binet-type formulas. Although it is a straightforward exercise using mathematical induction to prove that Binet's formulas do produce the sequences desired, it is not obvious as to how such Binet formulas are obtained. An analysis of the Binet-type formula shows why it works for a recursive sequence and also how other recursive sequences can be developed.

Generalized Fibonacci Sequences

Generalized Fibonacci sequences G_n are usually defined recursively [2, p. 1] as a sequence of positive integers such that for positive integers a, b, c, and d,

$$G_1 = a,\ G_2 = b, \text{ and } G_{n+1} = cG_{n-1}+dG_n \text{ for all } n \geq 2. \quad (1)$$

If a=b=c=d=1, the sequence is the Fibonacci sequence $\{F_n\}$. If a=c=d=1 and b=3, the sequence is the Lucas sequence $\{L_n\}$. Fibonacci and Lucas sequences can be defined explicitly using Binet's formulas:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{for all } n \geq 1, \qquad (2)$$

$$L_n = \alpha^n + \beta^n \quad \text{for all } n \geq 1, \qquad (3)$$

where $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$ [3, p. 272-3].

Fibonacci-type sequences arise in many places [4, p.283]. In their discovery of the incommensurability of the diagonal of a square to the side of the square, the Pythagoreans used integer sequences of lengths in the constructions of the side and diagonal lengths [5, p. 2]. Serendipitously, these sequences are examples of Fibonacci-type sequences, and the side numbers $\{a_n\}$ and the diagonal numbers $\{d_n\}$ can be defined recursively using (1) above: for $\{a_n\}$, let a=c=1 and b=d=2 and for
$\{d_n\}$, let a=c=1, b=3, and d=2. Thus, $\{a_n\}$ is defined by
$a_1=1$, $a_2=2$, $a_{n+1} = 2a_n + a_{n-1}$ for all $n \geq 2$, (4)

and the sequence $\{d_n\}$ is defined by

$$d_1=1,\ d_2=3,\ d_{n+1}= 2d_n + d_{n-1} \text{ for all } n \geq 2, \quad (5).$$

Explicit representations of the sequences can be found using “Binet-type” formulas.

The Binet Formula

Suppose $\alpha = 1 + \sqrt{k}$ and $\beta = 1 - \sqrt{k}$, where k is a positive integer. If we assume that some integer sequence $\{x_n\}$ is defined by a Binet formula using these numbers, then one possibility is that

$$x_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{for all } n \geq 1, \qquad (5)$$

which in this case would mean that

$$x_n = \frac{(1+\sqrt{k})^n - (1-\sqrt{k})^n}{2\sqrt{k}} \quad \text{for all } n \geq 1. \quad (6)$$

An explanation needs to be given to justify that the formula given in (6) will produce an integer for each positive integer k and for each positive integer n.

Using the binomial theorem to expand the numerator in (6), we have

$$(1+\sqrt{k})^n - (1-\sqrt{k})^n = \sum_{i=0}^{n} \binom{n}{i} (\sqrt{k})^i - \sum_{i=0}^{n} \binom{n}{i} (-\sqrt{k})^i$$

In the preceding sum, when i is even, since $(\sqrt{k})^i = (-\sqrt{k})^i$, the even terms vanish. Thus, for all odd integers i, $1 \leq i \leq n$, the preceding sum becomes

$(1+\sqrt{k})^n - (1-\sqrt{k})^n$

$= \sum_{i\ odd}[\binom{n}{i}(\sqrt{k})^i - \binom{n}{i}(-\sqrt{k})^i]$

$= \sqrt{k}\sum_{i\ odd}[\binom{n}{i}(\sqrt{k})^{i-1} + \binom{n}{i}(-\sqrt{k})^{i-1}]$

$= \sqrt{k}\sum_{i\ odd}[\binom{n}{i}(\sqrt{k})^{i-1} + \binom{n}{i}(\sqrt{k})^{i-1}]$, since i-1 is even,
$(\sqrt{k})^{i-1} = (-\sqrt{k})^{i-1}$;

$= 2\sqrt{k}\sum_{i\ odd}\binom{n}{i}(\sqrt{k})^{i-1}$.

Hence, the formula given in (6) becomes

$$x_n = \frac{2\sqrt{k}\sum_{i\ odd}\binom{n}{i}(\sqrt{k})^{i-1}}{2\sqrt{k}} = \sum_{i\ odd}\binom{n}{i}(\sqrt{k})^{i-1}$$

which is clearly an integer. Notice that if

$$x_n = \frac{(m+\sqrt{k})^n - (m-\sqrt{k})^n}{2\sqrt{k}}$$

for any positive integer m, one can show via a similar argument that the formula generates an integer sequence. One can also use strong mathematical induction on n to establish the result for any fixed positive integers m and k.

Using the formula given in (6) one discovers that the first four terms of sequence $\{x_n\}$ are $x_1 = 1$, $x_2 = 2$, $x_3 = 3+k$, and $x_4 = 4+4k$. If one assumes that there is a recursive definition for the sequence $\{x_n\}$, so that $\{x_n\}$ is a Fibonacci-type sequence, we need only solve for c and d in (1). This leads to the fact that

$$c + 2d = 3 + k, \text{ and}$$
$$2c + d(3+k) = 4 + 4k. \quad (7)$$

Solving the system of equations for c and d leads to the result that
d=2 and c= k-1. So $\{x_n\}$ is defined recursively as follows:
$x_1 = 1$, $x_2 = 2$, $x_{n+1} = 2x_n + (k-1)x_{n-1}$ for all $n \geq 1$.

Thus, if we let k=2 it follows that the Binet formula for the integer sequence of side number in (4) is given by

$$a_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}} \quad \text{for all } n \geq 1, (8)$$

a result that is obtained by substituting 2 for k in the equation in (6).

Another possibility for constructing a Binet formula is to change the values used for α and β. For example, if

$\alpha = \dfrac{1+\sqrt{k}}{2}$, $\beta = \dfrac{1-\sqrt{k}}{2}$, and these values are substituted into (5), it follows that a new sequence $\{y_n\}$ is formed and $\{y_n\}$ has the property that

$$y_n = \frac{x_n}{2^{n-1}} \text{ for all } n \geq 1. \quad (9)$$

Using the result in (9) it can easily be found that the first four terms of $\{y_n\}$ are $y_1 = 1$,

$y_2 = 1, y_3 = \dfrac{3+k}{4}$, and $y_4 = \dfrac{1+k}{2}$. Now it becomes apparent that $\{y_n\}$ will be an integer sequence only for certain values of k, namely $k = 4m + 1$, where m is some positive integer. If a recursive Fibonacci-type sequence for this choice of α and β is assumed then the values of c and d in (1) are $d = 1$ and $c = \frac{1}{4}(k-1)$. Thus, one can see that the choice of k for the Fibonacci sequence must be the integer 5 so that $c = 1$ as well as $d = 1$. This produces the well-known formula given in (2).

Another choice for constructing Binet formulas would be to use the addition form

$$x_n = \alpha^n + \beta^n \text{ for all } n \geq 1. \quad (10)$$

If $\alpha = 1 + \sqrt{k}$ and $\beta = 1 - \sqrt{k}$, then $x_1 = 2$, $x_2 = 2+2k$, $x_3 = 2+6k$, and $x_4 = 2 + 12k + 2k^2$.

Because all the terms have a factor of 2, the form in (10) is changed to

$$x_n = \tfrac{1}{2}(\alpha^n + \beta^n) \text{ for all } n \geq 1. \quad (11)$$

Now the first four terms of the sequence will be $x_1 = 1$, $x_2 = 1+k$, $x_3 = 1+3k$, and $x_4 = 1 + 6k + k^2$. Assuming a recursive, Fibonacci-type sequence, one can use these first four terms and solve for the positive integers c and d in (1):

$$c+d(1+k) = 1 + 3k$$
$$c(1+k)+d(1+3k) = 1 + 6k + k^2.$$

Solving these equations simultaneously for c and d reveals that $c = k-1$ and $d = 2$. If $k=2$, then the first four terms of the sequence will be 1,3,7, and 17. These are the first four terms of the diagonal number sequence in (5). Since the diagonal number sequence is a recursive Fibonacci-type sequence, the Binet formula for the diagonal number sequence in (5) is

$$d_n = \tfrac{1}{2}\left[(1+\sqrt{2})^n + (1-\sqrt{2})^n\right] \text{ for all } n \geq 1. \quad (12)$$

If, on the other hand, one's choices for α and β are $\alpha = \frac{1}{2}(1+\sqrt{k})$ and $\beta = \frac{1}{2}(1-\sqrt{k})$, one can use the result in (9) to find that the first four terms of the integer sequence are 1, $\frac{1}{2}(1+k)$, $\frac{1}{4}(1+3k)$, and $\frac{1}{8}(1+6k+k^2)$. Again, only certain positive integers k will yield an integer sequence, namely $k = 4m+1$, where m is any positive integer. Solving for c and d in (1) results in $c = \frac{1}{4}(k-1)$ and $d = 1$. If $k=5$, then one gets that $c=1$ and $d=1$, and the well-known Binet formula for the Lucas sequence in (3) is obtained. For other choices of k= 4m +1 such as k=13 or 17, other sequences are obtained.1

Connections between the Side and Diagonal Numbers

Just as there are many connections between the Fibonacci and Lucas numbers [1, pp. 67-72] the same is true for the side and diagonal numbers. For example,

$$2a_nd_n = a_{2n} \text{ for all } n \geq 1. \quad (13)$$

Although a proof by induction is not difficult, a simple algebraic proof can be given using the Binet formula for the side numbers in (8) and that for the diagonal numbers in (12). Using those two formulas the proof goes as follows: For all n $\geq$ 1,

$$2a_nd_n = 2\left[\frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}\right]\left[\frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n}{2}\right] = \frac{(1+\sqrt{2})^{2n} - (1-\sqrt{2})^{2n}}{2\sqrt{2}} = a_{2n},$$

and the proof of the result in (13) is complete. Just as the Binet formulas aid in establishing proofs of connections between the Fibonacci and Lucas sequences, they play the same role in establishing connections between the side and diagonal numbers**.**

Constructing Recursive Fibonacci-type Sequences and their Binet Formulas

Using the same proof techniques as before, we can easily obtain the following
Theorem: If m and k are positive integers such that k $\geq$ $m^2 + 1$ and

$$y_n = \frac{(m+\sqrt{k})^n - (m-\sqrt{k})^n}{2\sqrt{k}}$$

for all n $\geq$ 1, and $\{x_n\}$ is a recursive Fibonacci-type sequence defined as $x_1 = 1$, $x_2 = 2m$, and $x_{n+1} = (k - m^2)\, x_{n-1} + (2m)x_n$ for all n $\geq$ 2, then $\{x_n\}=\{y_n\}$.

Note: Before proving the theorem, this note of clarification is given. The proof of the theorem consists of two parts. First, it is proved that the Binet formula generates the sequence $\{x_n\}$. Then it is proved that the sequence $\{y_n\}$ must be the same sequence as $\{x_n\}$; that is $\{y_n\}= \{x_n\}$.
Proof of Theorem: Suppose $\{x_n\}$ is a recursive Fibonacci-type sequence defined in the statement of the theorem. The proof is by induction. For the case n=1,

$$x_1 = 1 = \frac{(m+\sqrt{k})^1 - (m-\sqrt{k})^1}{2\sqrt{k}}.$$

If we assume that

$$x_n = \frac{(m+\sqrt{k})^n - (m-\sqrt{k})^n}{2\sqrt{k}}$$

for all positive integers n such that $1 \le n \le j$ where j is some positive integer, then

$$x_{j+1} = (k - m^2)\, x_{j-1} + (2m)x_j = (k - m^2)\left[\frac{(m+\sqrt{k})^{j-1} - (m-\sqrt{k})^{j-1}}{2\sqrt{k}}\right] + (2m)\left[\frac{(m+\sqrt{k})^{j} - (m-\sqrt{k})^{j}}{2\sqrt{k}}\right]$$

$$= \frac{(m+\sqrt{k})^{j-1}}{2\sqrt{k}}[(k - m^2) + (m+\sqrt{k})(2m)] + [(k - m^2) + (m-\sqrt{k})(2m)]$$

$$= \frac{(m+\sqrt{k})^{j-1}}{2\sqrt{k}}[k + 2m\sqrt{k} + m^2] - \frac{(m-\sqrt{k})^{j-1}}{2\sqrt{k}}[k - 2m\sqrt{k} + m^2]$$

$$= \frac{(m+\sqrt{k})^{j-1}}{2\sqrt{k}}(m+\sqrt{k})^2 - \frac{(m-\sqrt{k})^{j-1}}{2\sqrt{k}}(m - \sqrt{k})^2$$

$$= \frac{(m+\sqrt{k})^{j+1}}{2\sqrt{k}} - \frac{(m-\sqrt{k})^{j+1}}{2\sqrt{k}} = \frac{(m+\sqrt{k})^{j+1} - (m-\sqrt{k})^{j+1}}{2\sqrt{k}}.$$

Thus, the preceding part of the proof by mathematical induction establishes the fact that this Binet formula generates the recursive sequence $\{x_n\}$. This induction proof shows that the recursive sequence $\{x_n\}$defined in this theorem has this Binet formula.

Now suppose we consider the sequence $\{y_n\}$ this Binet formula produces. In the induction proof it was verified that when n =1, the Binet formula generates the first term of the defined recursive sequence $\{x_n\}$, namely 1. Likewise, for n=2 the Binet formula generates the second term, namely 2m. In the induction step of the proof above it was shown that

$$(k-m^2)\left[\frac{(m+\sqrt{k})^{n-1} - (m-\sqrt{k})^{n-1}}{2\sqrt{k}}\right] + (2m)\left[\frac{(m+\sqrt{k})^{n} - (m-\sqrt{k})^{n}}{2\sqrt{k}}\right]$$

$= \dfrac{(m+\sqrt{k})^{n+1} - (m-\sqrt{k})^{n+1}}{2\sqrt{k}}$, and this result holds for all $n \ge 1$. So this would say that $(k - m^2)\, y_{n-1} + (2m)\, y_n = y_{n+1}$. But this is precisely the way the recursive sequence $\{x_n\}$ was defined. So the sequence the Binet formula produces is a recursive Fibonacci-type sequence defined in the same way $\{x_n\}$ is defined, and hence, $\{x_n\} = \{y_n\}$. So the only sequence the Binet formula produces is the given recursive sequence. (Note: We could also argue that $\{x_n\}$ is the only sequence that the Binet formula generates because $\dfrac{(m+\sqrt{k})^{n} - (m-\sqrt{k})^{n}}{2\sqrt{k}}$ is a function of n, and thus can generate only one value for each value of n, and so the Binet formula can generate only one sequence, and that sequence from the induction argument is $\{x_n\}$.)

By using various choices for m and k, one can generate many Fibonacci-type sequences and their corresponding Binet formulas. See table I for some examples. In this table the values of m are the first 15 positive integers and $k = 3(m^2 + 1)$.

Conclusion

Recursive Fibonacci-type sequences have Binet-type formulas that can aid in constructing simple algebraic proofs of many properties of the sequences. Explorations with Binet formulas remove some of the mystery as to how these formulas arise. Side and diagonal numbers have yet many connections to be discovered. Binet formulas will be extremely beneficial in constructing proofs of such connections. Research of this nature can provide questions for students to investigate, and thus can serve as resources for mathematics teachers who are looking for sequences that are accessible to students with limited background and preparation.

† *Homer W. Austin, Ph.D.*, Salisbury University, Maryland, USA
‡ *Jathan W. Austin*, University of Delaware, Delaware, USA

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Table I: Recursive Fibonacci-type Sequences

m	k	c	d	X_1	X_2	X_3	X_4	Definition of Sequence $X_{n+1} = c\, X_{n-1} + d\, X_n$	Binet Formula $X_n = \dfrac{(m+\sqrt{k})^n - (m-\sqrt{k})^n}{2\sqrt{k}}$
1	6	5	2	1	2	9	28	$X_1 = 1, X_2 = 2$ $X_{n+1} = 5\, X_{n-1} + 2\, X_n$	$X_n = \dfrac{(1+\sqrt{6})^n - (1-\sqrt{6})^n}{2\sqrt{6}}$
2	15	11	4	1	4	27	152	$X_1 = 1, X_2 = 4$ $X_{n+1} = 11\, X_{n-1} + 4\, X_n$	$X_n = \dfrac{(2+\sqrt{15})^n - (2-\sqrt{15})^n}{2\sqrt{15}}$
3	30	21	6	1	6	57	468	$X_1 = 1, X_2 = 6$ $X_{n+1} = 21 X_{n-1} + 6\, X_n$	$X_n = \dfrac{(3+\sqrt{30})^n - (3-\sqrt{30})^n}{2\sqrt{30}}$
4	51	35	8	1	8	99	1072	$X_1 = 1, X_2 = 8$ $X_{n+1} = 35\, X_{n-1} + 8 X_n$	$X_n = \dfrac{(4+\sqrt{51})^n - (4-\sqrt{51})^n}{2\sqrt{51}}$
5	78	53	10	1	10	153	2060	$X_1 = 1, X_2 = 10$ $X_{n+1} = 53 X_{n-1} + 10\, X_n$	$X_n = \dfrac{(5+\sqrt{78})^n - (5-\sqrt{78})^n}{2\sqrt{78}}$
6	111	75	12	1	12	219	3528	$X_1 = 1, X_2 = 12$ $X_{n+1} = 75\, X_{n-1} + 12\, X_n$	$X_n = \dfrac{(6+\sqrt{111})^n - (6-\sqrt{111})^n}{2\sqrt{111}}$
7	150	101	14	1	14	297	5572	$X_1 = 1, X_2 = 14$ $X_{n+1} = 101 X_{n-1} + 14 X_n$	$X_n = \dfrac{(7+\sqrt{150})^n - (7-\sqrt{150})^n}{2\sqrt{150}}$
8	195	131	16	1	16	387	8288	$X_1 = 1, X_2 = 16$ $X_{n+1} = 131 X_{n-1} + 16 X_n$	$X_n = \dfrac{(8+\sqrt{195})^n - (8-\sqrt{195})^n}{2\sqrt{195}}$
9	246	165	18	1	18	489	11772	$X_1 = 1, X_2 = 18$ $X_{n+1} = 165\, X_{n-1} + 18 X_n$	$X_n = \dfrac{(9+\sqrt{246})^n - (9-\sqrt{246})^n}{2\sqrt{246}}$
10	303	203	20	1	20	603	16120	$X_1 = 1, X_2 = 20$ $X_{n+1} = 203 X_{n-1} + 20 X_n$	$X_n = \dfrac{(10+\sqrt{303})^n - (10-\sqrt{303})^n}{2\sqrt{303}}$
11	366	245	22	1	22	729	21428	$X_1 = 1, X_2 = 22,$ $X_{n+1} = 245\, X_{n-1} + 22\, X_n$	$X_n = \dfrac{(11+\sqrt{366})^n - (11-\sqrt{366})^n}{2\sqrt{366}}$
12	435	291	24	1	24	867	27792	$X_1 = 1, X_2 = 24$ $X_{n+1} = 291\, X_{n-1} + 24 X_n$	$X_n = \dfrac{(12+\sqrt{435})^n - (12-\sqrt{435})^n}{2\sqrt{435}}$
13	510	341	26	1	26	1017	35308	$X_1 = 1, X_2 = 26$ $X_{n+1} = 341\, X_{n-1} + 26\, X_n$	$X_n = \dfrac{(13+\sqrt{510})^n - (13-\sqrt{510})^n}{2\sqrt{510}}$
14	591	395	28	1	28	1179	44072	$X_1 = 1, X_2 = 28$ $X_{n+1} = 395 X_{n-1} + 28 X_n$	$X_n = \dfrac{(14+\sqrt{591})^n - (14-\sqrt{591})^n}{2\sqrt{591}}$
15	678	453	30	1	30	1353	54180	$X_1 = 1, X_2 = 30$ $X_{n+1} = 453\, X_{n-1} + 30\, X_n$	$X_n = \dfrac{(15+\sqrt{678})^n - (15-\sqrt{678})^n}{2\sqrt{678}}$