Separation in Homogeneous Continua

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Abstract

It is known that homogeneous continua cannot be separated by a single point. However, there is no guarantee in general that the subcontinua in such spaces have the same property. This paper establishes the fact that, in all but the most simple case, the nontrivial subcontinua with minimal finite boundary in a homogeneous continuum share the property of the entire space that they cannot be separated by a single point.

More specifically, if X is a homogeneous continuum, then no single point separates X. If the nontrivial subcontinua of X with minimal boundary contain precisely two boundary points, then these subcontinua can be separated by a single point. However, if these subcontinua have finite boundary containing more than two points, then like X itself, they cannot be separated by a single point.

Introduction

Due to some variation throughout the literature, basic definitions and notations are presented. Some preliminary results related to homogeneous continua and subcontinua with minimal boundary are developed to provide access to the main theorems. These results will depend on the work of Simmons [9] and Winton [12]. Some of the results of Simmons [9] will also be generalized. We begin with basic definitions and notations.

Definitions

Continua are sometimes defined to be compact, connected metric spaces. However, we will adopt the more general approach by defining a continuum to be a compact, connected, Hausdorff topological space. If H is a subset of a topological space X, then Int(H), Cl(H), and Bd(H) are the topological interior, closure, and boundary of H, respectively. A separation $A \mid B$ of a space X is a partition of X into nonempty relatively open sets A and B. A subset H of X separates X if and only if X is connected but X−H is not connected. In particular, a point $p \in X$ is a cut point of X if and only if X is connected but $X-\{p\}$ is not connected. That is, p is a cut point of X if and only if $\{p\}$ separates X. A point $p \in X$ is a noncut point of X if and only if p is not a cut point of X. If n is an integer greater than 1, X is a topological space, and $H \subseteq X$, then H is an n-pod of X if and only if H is a subcontinuum of X whose boundary contains precisely n points. Furthermore, n is the pod number of X if and only if X contains an n-pod but X contains no k-pod whenever k is an integer and $1 < k < n$. The pod number of X is denoted by P(X).

Preliminary Results

Lemma 1. Suppose X is a topological space and $A \subseteq B \subseteq C \subseteq X$.

(a) If A is open in C, then A is open in B.

(b) If A is closed in C, then A is closed in B.

(c) If A is open in B and B is open in C, then A is open in C.

(d) If A is closed in B and B is closed in C, then A is closed in C.

Proof.

(a) Since A is open in C then $A = O \cap C$ for some open set O in X. Therefore $A = A \cap B$ (since $A \subseteq B$) $= (O \cap C) \cap B = O \cap (C \cap B) = O \cap B$ (since $B \subseteq C$). Hence A is open in B.

(b) Since A is closed in C then $A = F \cap C$ for some closed set F in X. Therefore $A = A \cap B$ (since $A \subseteq B$) $= (F \cap C) \cap B = F \cap (C \cap B) = F \cap B$ (since $B \subseteq C$). Hence A is closed in B.

(c) Since A is open in B then $A = O \cap B$ for some open set O in X. Since B is open in C then $B = V \cap C$ for some open set V in X. Therefore $O \cap V$ is open in X. Furthermore, $A = O \cap B = O \cap (V \cap C) = (O \cap V) \cap C$. Hence A is open in C.

(d) Since A is closed in B then $A = F \cap B$ for some closed set F in X. Since B is closed in C then $B = G \cap C$ for some closed set G in X. Therefore $F \cap G$ is closed in X. Furthermore, $A = F \cap B = F \cap (G \cap C) = (F \cap G) \cap C$. Hence A is closed in C.

It is known that a homogeneous continuum cannot be separated by a single point. However, we present this result here since it is closely related to the main theorems of the paper.

Theorem 2. If X is a homogeneous continuum, then X cannot be separated by a single point. That is, X contains no cut points.

Proof. If $X = \{p\}$ is a trivial space then $X - \{p\} = \varnothing$. Thus there can be no separation of $X - \{p\}$, and so p is a noncut point.

On the other hand, if X is nontrivial, then X contains at least two noncut points p and q [11, p. 205, Theorem 28.8]. If $y \in X$ then there exists a homeomorphism $f: X \to X$ such that $f(p) = y$ since X is homogeneous. Since p is a noncut point of X and f is a homeomorphism, then y is also a noncut point of X. Therefore each point in X is a noncut point.

Thus in either case X contains no cut points. Hence X cannot be separated by a single point.

Simmons showed that if X is a homogeneous continuum, H is a 2-pod in X, $p \in X$, and $A|B$ is a separation of $H - \{p\}$, then A and B each contain a point of $\mathrm{Bd}(H)$ [9, Lemma 3]. Lemma 3 generalizes this result to include all n-pods in X.

Lemma 3. Suppose X is a homogeneous continuum, H is an n-pod in X, $p \in X$, and $A|B$ is a separation of $H-\{p\}$. Then A and B each contain a point of $\mathrm{Bd}(H)$.

Proof. Suppose $A \cap \mathrm{Bd}(H) = \varnothing$. If $p \notin H$ then $H-\{p\} = H$. However, $H-\{p\}$ is not connected since $A|B$ is a separation of $H-\{p\}$, while H is connected. This is a contradiction, and so $p \in H$.

Define $W = B \cup (X-H)$. We now show that $\{A,B,X-H\}$ and $\{A,W\}$ are partitions of $X-\{p\}$. Clearly $A \neq \varnothing$ and $B \neq \varnothing$ since $A|B$ is a separation of $H-\{p\}$. If $X-H = \varnothing$, then $H = X$. Then $A|B$ is a separation of $H-\{p\} = X-\{p\}$, so that p is a cut point of X. This contradicts Theorem 2, and so $X-H \neq \varnothing$. Since $A|B$ is a separation of $H-\{p\}$, then $A \cap B = \varnothing$. Furthermore, $A \subseteq H$ and $B \subseteq H$, so that $A \cap (X-H) = \varnothing = B \cap (X-H)$. Finally, since it was shown above that $p \in H$, then $X-\{p\} = (H-\{p\}) \cup (X-H) = A \cup B \cup (X-H)$. Thus $\{A,B,X-H\}$ is a partition of $X-\{p\}$. Since $W = B \cup (X-H)$, then $\{A,W\}$ is also a partition of $X-\{p\}$.

To show that A is open in $X-\{p\}$, recall that it was shown above that $A \cap (X-H) = \varnothing$. Furthermore, $A \cap \mathrm{Bd}(H) = \varnothing$ by the assumption above. Therefore $\varnothing = \varnothing \cup \varnothing = [A \cap (X-H)] \cup [A \cap \mathrm{Bd}(H)] = A \cap [(X-H) \cup \mathrm{Bd}(H)] = A \cap [(X-H) \cup \mathrm{Bd}(X-H)] = A \cap \mathrm{Cl}(X-H)$ ([8, p. 87, no. 12],[11, p. 28, Theorem 3.14(a)]), and so A contains no limit points of $X-H$ [10, p. 96, Theorem D(1)]. Furthermore, since $A|B$ is a separation of $H-\{p\}$, then A contains no limit points of B. Thus A contains no limit points of $B \cup (X-H) = W$. Since $\{A,W\}$ is a partition of $X-\{p\}$, then W must contain its own limit points in $X-\{p\}$. Thus W is closed in $X-\{p\}$ [10, p. 96, Theorem D(2)], and so $A = (X-\{p\})-W$ is open in $X-\{p\}$.

To show that W is open in $X-\{p\}$, note that since H is compact and X is Hausdorff, then H is closed in X ([1, p. 81, Corollary 5.13],[3, p. 165, Theorem 6.4]). Thus H contains its own limit points in X [10, p. 96, Theorem D(2)], and so $X-H$ contains no limit points of H. Therefore $X-H$ contains no limit points of A since $A \subseteq H$. Since $A|B$ is a separation of $H-\{p\}$, then B contains no limit points of A either. Thus $W = B \cup (X-H)$ contains no limit points of A. Since $\{A,W\}$ is a partition of $X-\{p\}$, then A must contain its own limit points in $X-\{p\}$. Thus A is closed in $X-\{p\}$ [10, p. 96, Theorem D(2)], and so $W = (X-\{p\})-A$ is open in $X-\{p\}$.

Therefore $A|W$ is a separation of $X-\{p\}$, so that p is a cut point of the homogeneous continuum X. However, this contradicts Theorem 2, and so $A \cap \mathrm{Bd}(H) \neq \varnothing$. Similarly $B \cap \mathrm{Bd}(H) \neq \varnothing$.

Simmons showed that if X is a homogeneous continuum, H is a 2-pod in X, $p \in X$, and $A|B$ is a separation of $H-\{p\}$, then $A \cup \{p\}$ and $B \cup \{p\}$ are 2-pods in X. Furthermore, there exist $r,t \in X$ such that $\mathrm{Bd}(H) = \{r,t\}$, $\mathrm{Bd}(A \cup \{p\}) = \{r,p\}$

and $Bd(B\cup\{p\}) = \{t,p\}$ [9, Lemma 5]. Lemma 4 generalizes this result to include all n-pods in X.

Lemma 4. Suppose X is a homogeneous continuum, H is an n-pod in X, $p\in X$, and $A|B$ is a separation of $H-\{p\}$. Then $A\cup\{p\}$ and $B\cup\{p\}$ are subcontinua of X with $Bd(A\cup\{p\}) = [A\cap Bd(H)]\cup\{p\}$ and $Bd(B\cup\{p\}) = [B\cap Bd(H)]\cup\{p\}$.

Proof. We will begin by showing that $A\cup\{p\}$ is a subcontinuum of X. Define $K = A\cup\{p\}$ and $W = B\cup(X-H)$. It was shown in Lemma 3 that $\{A,W\}$ is a partition of $X-\{p\}$, so that $\{K,W\}$ is a partition of X, and so $K = X-W$. It was also shown in Lemma 3 that W is open in $X-\{p\}$. Since X is Hausdorff then $\{p\}$ is closed in X [5, p. 64, Corollary 3.12], and so $X-\{p\}$ is open in X. Since W is open in $X-\{p\}$ and $X-\{p\}$ is open in X, then W is open in X by Lemma 1(c). Thus $K = X-W$ is closed in X. Since K is a closed subset of the compact space X, then K is compact ([8, p. 162, Theorem 2.11],[10, p. 111, Theorem A]). Furthermore, since H and $\{p\}$ are connected subsets of X and $A|B$ is a separation of $H-\{p\}$, then $K = A\cup\{p\}$ is connected [12, Lemma 1]. Hence $A\cup\{p\} = K$ is a subcontinuum of X.

To show that $Bd(A\cup\{p\}) = [A\cap Bd(H)]\cup\{p\}$, recall that it was shown above that K is closed in X. Therefore $Bd(K) \subseteq K$ [3, p. 105, Theorem 4.5(6)].

Furthermore, to show that $K-([A\cap Bd(H)]\cup\{p\})$ contains no points of Bd(K), note that $B\cup\{p\}$ is closed in X by an argument similar to that above for $K = A\cup\{p\}$. Therefore $O = X-(B\cup\{p\})$ is open in X. Furthermore, $H-Bd(H) = Int(H)$ ([6, p. 46, Theorem 10],[11, p. 28, Theorem 3.14(b)]) is open in X as well. Define $V = O\cap Int(H)$, which is open in X. Since $A|B$ is a separation of $H-\{p\}$, then $\{A,B\}$ is a partition of $H-\{p\}$, and so $\{A,B,\{p\}\}$ is a partition of H since it was shown in Lemma 3 that $p\in H$.

Suppose $q\in K-([A\cap Bd(H)]\cup\{p\})$. Then $q\in K = A\cup\{p\}$ and $q\notin[A\cap Bd(H)]\cup\{p\}$, so that $q\in A\cup\{p\}$, $q\notin A\cap Bd(H)$, and $q\notin\{p\}$. Since $q\in A\cup\{p\}$ but $q\notin\{p\}$ then $q\in A$. Since $q\in A$ but $q\notin A\cap Bd(H)$ then $q\notin Bd(H)$. Therefore $q\in A$ and $q\notin Bd(H)$. Since $q\in A$ then $q\notin B\cup\{p\}$ since $\{A,B,\{p\}\}$ is a partition of H, and so $q\in X-(B\cup\{p\}) = O$. Furthermore, since $q\in A \subseteq H$ and $q\notin Bd(H)$ then $q\in H-Bd(H) = Int(H)$ ([6, p. 46, Theorem 10],[11, p. 28, Theorem 3.14(b)]). Therefore $q\in O\cap Int(H) = V$.

Finally, if $x\in V = O\cap Int(H)$ then $x\in O$ and $x\in Int(H)$. Since $x\in Int(H)$ then $x\in H$ since $Int(H) \subseteq H$. Furthermore, since $x\in O = X-(B\cup\{p\})$ then $x\notin B\cup\{p\}$. Thus $x\in H-(B\cup\{p\}) = A$ since $\{A,B,\{p\}\}$ is a partition of H. Therefore $V \subseteq A$.

Thus if $q\in K-([A\cap Bd(H)]\cup\{p\})$, then there is an open set V in X such that $q\in V \subseteq A$, and so $q\in Int(A)$. Furthermore, since $A \subseteq K$, then $Int(A) \subseteq Int(K)$ ([3, p. 103, Theorem 4.3(3)],[7, p. 90, Theorem 1(iv)]). Therefore $q\in Int(A) \subseteq Int(K) = K-Bd(K)$ ([6, p. 46, Theorem 10],[11, p. 28, Theorem 3.14(b)]), so that $q\notin Bd(K)$.

Since $Bd(K) \subseteq K$ and $q\notin Bd(K)$ for each $q\in K-([A\cap Bd(H)]\cup\{p\})$, then $Bd(K) \subseteq [A\cap Bd(H)]\cup\{p\}$.

Conversely, to show that $A \cap Bd(H) \subseteq Bd(K)$, suppose that $q \in A \cap Bd(H)$, O is open in X, and $q \in O$. Since $A|B$ is a separation of $H-\{p\}$, then A is open in $H-\{p\}$. Thus $A = V \cap (H-\{p\})$ for some open set V in X. Define $U = O \cap V$, which is open in X. Since $q \in O$ and $q \in A \cap Bd(H) \subseteq A = V \cap (H-\{p\}) \subseteq V$, then $q \in O \cap V = U$. Since $q \in Bd(H)$, U is open in X, and $q \in U$, then $U \cap H \neq \varnothing$ and $U \cap (X-H) \neq \varnothing$, and so there exist $c \in U \cap H$ and $d \in U \cap (X-H)$. Furthermore, it was shown in Lemma 3 that $p \in H$, so that $H = (H-\{p\}) \cup \{p\}$. Therefore $c \in U \cap H = (O \cap V) \cap H = (O \cap V) \cap [(H-\{p\}) \cup \{p\}] = [(O \cap V) \cap (H-\{p\})] \cup [(O \cap V) \cap \{p\}] = [O \cap (V \cap (H-\{p\}))] \cup [O \cap (V \cap \{p\})] = [O \cap A] \cup [O \cap (V \cap \{p\})] \subseteq [O \cap A] \cup [O \cap \{p\}] = O \cap (A \cup \{p\}) = O \cap K$. Furthermore, since $K \subseteq H$ then $X-H \subseteq X-K$. Thus $d \in U \cap (X-H) \subseteq U \cap (X-K) = O \cap V \cap (X-K) \subseteq O \cap (X-K)$. Hence $O \cap K \neq \varnothing$ and $O \cap (X-K) \neq \varnothing$ for each open set O in X containing q, so that $q \in Bd(K)$. Therefore $q \in Bd(K)$ for each $q \in A \cap Bd(H)$, and so $A \cap Bd(H) \subseteq Bd(K)$.

Finally, to show that $p \in Bd(K)$, suppose O is open in X and $p \in O$. Assume that $O \cap A = \varnothing$. Define $V = O \cap H$ and $U = B \cup V$. Since $A|B$ is a separation of $H-\{p\}$, then $A \neq \varnothing$, $B \neq \varnothing$, and A is open in $H-\{p\}$. Therefore $U \neq \varnothing$ as well since $B \subseteq U$.

Since X is Hausdorff then $\{p\}$ is closed in X [5, p. 64, Corollary 3.12], so that $X-\{p\}$ is open in X. Thus $H-\{p\} = H \cap (X-\{p\})$ is open in H. Since A is open in $H-\{p\}$ and $H-\{p\}$ is open in H, then A is open in H by Lemma 1(c). Similarly B is open in H. Furthermore, since O is open in X and $V = O \cap H$, then V is open in H. Therefore $U = B \cup V$ is open in H. Thus A and U are nonempty relatively open sets in H.

Then $A \cap U = A \cap (B \cup V) = (A \cap B) \cup (A \cap V) = \varnothing \cup (A \cap V) = A \cap V = A \cap (O \cap H) = (A \cap O) \cap H = \varnothing \cap H$ (by the assumption above that $O \cap A = \varnothing$) $= \varnothing$. Furthermore, since $p \in O$ and it was shown in Lemma 3 that $p \in H$, then $p \in O \cap H = V$. Thus $A \cup U = A \cup (B \cup V) = (A \cup B) \cup V = (H-\{p\}) \cup V = H$ (since $p \in V$ and $V = O \cap H \subseteq H$).

Thus $A|U$ is a separation of the connected set H. This is a contradiction, and so $O \cap A \neq \varnothing$. Similarly $O \cap B \neq \varnothing$. Thus there exist $c \in O \cap A$ and $d \in O \cap B$. Then $c \in O \cap A \subseteq O \cap K$. Furthermore, since $A|B$ is a separation of $H-\{p\}$, then $A \cap B = \varnothing$ and $p \notin B$. Therefore $B \subseteq X-(A \cup \{p\}) = X-K$. Thus $d \in O \cap B \subseteq O \cap (X-K)$. Hence $O \cap K \neq \varnothing$ and $O \cap (X-K) \neq \varnothing$ for each open set O in X containing p, so that $p \in Bd(K)$.

Since $A \cap Bd(H) \subseteq Bd(K)$ and $p \in Bd(K)$, then $[A \cap Bd(H)] \cup \{p\} \subseteq Bd(K)$. Hence $Bd(A \cup \{p\}) = Bd(K) = [A \cap Bd(H)] \cup \{p\}$. In a similar manner, $B \cup \{p\}$ is a subcontinuum of X with $Bd(B \cup \{p\}) = [B \cap Bd(H)] \cup \{p\}$.

Having completed the preliminary material, we are now prepared to present the main results of the paper in two theorems. Together these results completely characterize the conditions under which a homogeneous continuum X with pod number $P(X) = n$ will contain n-pods which can be separated by a single point in X. Theorem 5 also identifies precisely which points in X separate the n-pods of X. The results of Theorem 5 and Theorem 6 are then combined in a corollary that follows.

Main Theorems

Simmons showed that if X is a homogeneous continuum, $P(X) = 2$, H is a 2-pod in X, $p\in X$, and $\{p\}$ separates H, then $p\in \mathrm{Int}(H)$ [9, Lemma 3]. It is not explicitly stated in this result that $P(X) = 2$. However, since it is assumed that X contains a 2-pod, then $P(X) \le 2$. Furthermore, by the results of [12, Corollary 4] and Theorem 2 of this paper, $P(X) = S(X) \ge 2$. Hence the hypothesis of [9, Lemma 3] implies that $P(X) = 2$. Theorem 5 extends this result by establishing the converse that if $p\in \mathrm{Int}(H)$, then $\{p\}$ separates H. Furthermore, it is verified that the result of Theorem 5 is not vacuously true by showing that $\mathrm{Int}(H) \ne \varnothing$.

Theorem 5. Suppose X is a homogeneous continuum, $P(X) = 2$, and H is a 2-pod in X. Then $\mathrm{Int}(H) \ne \varnothing$ and for each point $p\in X$, $\{p\}$ separates H if and only if $p\in \mathrm{Int}(H)$.

Proof. Since H is a 2-pod in X then $\mathrm{Bd}(H) = \{r,t\}$ for some $r,t\in X$. Then $\{r,t\}$ separates X and $\mathrm{Int}(H)\,|\,(X-H)$ is a separation of $X-\{r,t\}$ [12, Lemma 3]. Thus $\mathrm{Int}(H) \ne \varnothing$ and $X-H \ne \varnothing$. Since $X-H \ne \varnothing$ then there exists some $q\in X-H$.

If $p\in \mathrm{Int}(H)$, then $\{p,q\}$ separates X since $\{r,t\}$ separates X [9, main theorem]. Therefore $\{p\}$ separates H [9, Lemma 4]. Conversely, if $p\in X$ and $\{p\}$ separates H, then $p\in \mathrm{Int}(H)$ [9, Lemma 3]. Hence $\mathrm{Int}(H) \ne \varnothing$ and for each point $p\in X$, $\{p\}$ separates H if and only if $p\in \mathrm{Int}(H)$.

Theorem 6. Suppose X is a homogeneous continuum, $P(X) = n \ge 3$, and H is an n-pod in X. Then no single point in X separates H.

Proof. Suppose $p\in X$.

Case 1: Suppose $p\in X-H$. Then $H-\{p\} = H$, which is connected. Thus $\{p\}$ does not separate H.

Case 2: Suppose $p\in \mathrm{Bd}(H)$. If $\{p\}$ separates H, then $H-\{p\}$ is not connected, so there is a separation $A\,|\,B$ of $H-\{p\}$. Therefore $A\cap \mathrm{Bd}(H) \ne \varnothing$ and $B\cap \mathrm{Bd}(H) \ne \varnothing$ by Lemma 3. Since H is an n-pod in X and $p\in \mathrm{Bd}(H)$, then there exist $\{x_i\}_{i=1}^{n} \subseteq X$ $(n \ge 3)$ and an integer r such that $\mathrm{Bd}(H) = \{x_i\}_{i=1}^{n}$, $1 < r < n$, $p = x_r$, $\{x_i\}_{i=1}^{r-1} \subseteq A$, and $\{x_i\}_{i=r+1}^{n} \subseteq B$.

By Lemma 4, $A\cup\{p\}$ is a subcontinuum of X with $\mathrm{Bd}(A\cup\{p\}) = [A\cap \mathrm{Bd}(H)]\cup\{p\} = \{x_i\}_{i=1}^{r-1}\cup\{x_r\} = \{x_i\}_{i=1}^{r}$, so that $|\mathrm{Bd}(A\cup\{p\})| = r$. Therefore $A\cup\{p\}$ is an r-pod in X, where $r < n$. This is a contradiction since $P(X) = n$. Hence $\{p\}$ does not separate H.

Case 3: Suppose $p \in \mathrm{Int}(H)$. If $\{p\}$ separates H, then $H-\{p\}$ is not connected, so there is a separation $A|B$ of $H-\{p\}$. Since $p \in \mathrm{Int}(H) = H-\mathrm{Bd}(H)$ ([6, p. 46, Theorem 10],[11, p. 28, Theorem 3.14(b)]), then $p \notin \mathrm{Bd}(H)$. Therefore $\mathrm{Bd}(H) \subseteq H-\{p\} = A \cup B$. Since $n \geq 3$, then either A or B contains at least two points of Bd(H). Without loss of generality, suppose B contains at least two points of Bd(H). Furthermore, $A \cap \mathrm{Bd}(H) \neq \varnothing$ and $B \cap \mathrm{Bd}(H) \neq \varnothing$ by Lemma 3. Since H is an n-pod in X, then there exist $\{x_i\}_{i=1}^{n} \subseteq X$ ($n \geq 3$) and an integer r such that $\mathrm{Bd}(H) = \{x_i\}_{i=1}^{n}$, $1 \leq r \leq n-2$, $\{x_i\}_{i=1}^{r} \subseteq A$, and $\{x_i\}_{i=r+1}^{n} \subseteq B$.

By Lemma 4, $A \cup \{p\}$ is a subcontinuum of X with $\mathrm{Bd}(A \cup \{p\}) = [A \cap \mathrm{Bd}(H)] \cup \{p\} = \{x_i\}_{i=1}^{r} \cup \{p\}$, so that $|\mathrm{Bd}(A \cup \{p\})| = r + 1$. Therefore $A \cup \{p\}$ is an (r+1)-pod in X, where $r + 1 \leq (n - 2) + 1 = n - 1 < n$. This is a contradiction since $P(X) = n$. Hence $\{p\}$ does not separate H.

In conclusion, since $\{\mathrm{Int}(H), \mathrm{Bd}(H), X-H\}$ is a partition of X ([2, p. 142, Theorem 30.2],[4, p. 72, Theorem 4.11(4)]), then the above three cases imply that no single point in X separates H.

Theorem 5 and Theorem 6 can be combined to address all possible cases relative to the pod number of a homogeneous continuum. The following result establishes this fact.

Corollary 7. Suppose X is a homogeneous continuum, $P(X) = n$ and H is an n-pod in X. Then H can be separated by a single point in X if and only if $n = 2$. Furthermore, if $n = 2$, then $\{p \in X \mid \{p\} \text{ separates } H\} = \mathrm{Int}(H) \neq \varnothing$.

Proof. By the definition of the pod number of X, $P(X) \geq 2$. If $n = 2$, then by Theorem 5 $\mathrm{Int}(H) \neq \varnothing$ and $\{p\}$ separates X if and only if $p \in \mathrm{Int}(H)$. If $n > 2$, then by Theorem 6 no point in X separates H. The result follows.

Concluding Remarks

If Case 3 of Theorem 6 is omitted, then the result provides an extension of the last part of [9, Lemma 3] from 2-pods specifically to n-pods in general. For if H is an n-pod in a homogeneous continuum X, $p \in X$, and $A|B$ is a separation of $H-\{p\}$, then $\{p\}$ separates H. Then the contrapositives of Cases 1 and 2 of Theorem 6 imply that $p \notin X-H$ and $p \notin \mathrm{Bd}(H)$. Since $\{\mathrm{Int}(H), \mathrm{Bd}(H), X-H\}$ is a partition of X ([2, p. 142, Theorem 30.2],[4, p. 72, Theorem 4.11(4)]), then $p \in \mathrm{Int}(H)$. However, this is a moot point since Case 3 of Theorem 6 establishes that no point $p \in \mathrm{Int}(H)$ can separate H.

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References

[1] Cain, G. L., *Introduction to General Topology*, Addison Wesley Publishing Company, Reading, Massachusetts, 1994.

[2] Christie, D. E., *Basic Topology*, Macmillan Publishing Company, New York, 1976.

[3] Croom, F. H., *Principles of Topology*, Saunders College Publishing, Philadelphia, 1989.

[4] Dugundji, J., *Topology*, Allyn and Bacon, Inc., Boston, 1966.

[5] Hall, D. and Spencer, G., *Elementary Topology*, John Wiley & Sons, Inc., New York, 1955.

[6] Kelley, J. L., *General Topology*, D. Van Nostrand Company, Inc., Prineton, New Jersey, 1955.

[7] McCarty, G., *Topology*, Dover Publications, Inc., New York, 1967.

[8] Mendelson, B., *Introduction to Topology*, third edition, Dover Publications, Inc., New York, 1990.

[9] Simmons, F. W., *When Homogeneous Continua Are Hausdorff Circles (or Yes, We Hausdorff Bananas)*, Continua, Decompositions, and Manifolds, University of Texas Press (1980) pp. 62-73.

[10] Simmons, G. F., *Introduction to Topology and Modern Analysis*, McGraw Hill Book Company, Inc., San Francisco, 1963.

[11] Willard, S., *General Topology*, Addison Wesley Publishing Company, Inc., Reading, Massachusetts, 1970.

[12] Winton, R., *Complementary Subcontinua*, Journal of Mathematical Sciences and Mathematics Education, Vol. 2, No. 2, (2007) pp