Exact Sampling Distribution of Reliability Coefficient for Erlang – truncated Exponential Model

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Abstract

If the random variable X is representing the breaking strength of a given material and Y is a random variable representing the stress placed on that material then the probability that the material will survive is $P\left(X < Y\right)$. In this paper, we have developed the exact sampling distribution of reliability coefficient for Erlang–truncated Exponential model. Some distributional properties of the resulting distribution have been studied.

Introduction

Suppose that X and Y are the independent random variables with distribution functions $F\left(x\right)$ and $F\left(y\right)$ respectively. The estimation of the reliability from stress-strength has been of great importance in many physical situations e.g. Hall [6] provided an example of system application where the breakdown voltage Y of a capacitor must exceed the voltage output X of a power supply in order for a component to work properly. Weerahandi and Johnson [14] presented a rocket-motor experiment data where Y represents the chamber burst strength and X represents the operating pressure. Usually the studies have been carried out about the evaluation of P(X > Y), where X and Y are assumed to follow some known form of probability distributions. The practical application of stress-strength model does not confine only to engineering and military field but also more advances have been made in medical field in the last twenty years. A clinical trial is one of the fastest area under this topic.

Brinbaum and McCarty [3] initiated the work in this field. Church and Harris [4] were the first to formally use the term of stress – strength model for $P\left(X < Y\right)$. Mazumdar [12] has used the term of Inference theory for said probability. Estimation of reliability coefficient has been discussed by Hanagal [7] when (X_1, X_2) follow bivariate exponential models and X_3 follows independent exponential distribution. However a large number of papers are related to the probabilistic problems i.e. $R = P(X < Y)$ and developed a reliable and efficient estimator of this parameter based on the sample values with the assumption that X and Y are independent and both the random variables belong to the same family such as normal, weibull, exponentional etc. Wolfe and Hogg [15] had written a famous methodological note about the numerical values of $P(X<Y)$, which made more sense to the user. They provided a road-map towards the research which resulted in a number of papers on this topic. Bilikam [2] pointed out that the strength is "necessarily conditioned on the stress because the physical realization of strength is found only when stress is applied". He assumed that X and Y are independent and related time-dependent parameter. As time passes more developments were made on stress-strength model such as Hayter and Liu [8], Alam and Roohi [1], Kundu and Gupta [11], Khan and Islam [9], Sukuman *et al*. [13] etc. For comprehensive study readers may refer to the Kotz *et al*. [10].

In this paper we have derived and estimated the reliability coefficient for Erlang – truncated Exponential model and discussed some of its distributional properties. The organization of the paper is as follows: Reliability coefficient has been given in section 2. Section 3 and 4 contain the sampling distribution and moments of the distribution of the reliability coefficient respectively. Finally some concluding remarks are given in section 5.

The Reliability Coefficient

In this section we have derived the reliability coefficient for Erlang-truncated Exponential model given by El–Alosey [5]. We have also given an estimate of the resulting reliability coefficient. Suppose that the random variable X has Erlang-truncated Exponential distribution with density function:

$$f(x)=\beta_1\left(1-e^{-\lambda}\right)exp\left\{-\beta_1 x\left(1-e^{-\lambda}\right)\right\},$$

$$x>0\ ;\ \beta_1>0\ ;\ \lambda>0 \qquad (1)$$

Further suppose that random variable Y has the same distribution with density function:

$$f(y)=\beta_2\left(1-e^{-\lambda}\right)exp\left\{-\beta_2 y\left(1-e^{-\lambda}\right)\right\},$$

$$y>0\ ;\ \beta_2>0\ ;\ \lambda>0 \qquad (2)$$

Now, the reliability coefficient is defined as:

$$r=P(X<Y)=\int_0^\infty\int_0^y f(x,y)\,dxdy. \qquad (3)$$

Since X and Y are independent, so there joint distribution is:

$$f(x,y)=\beta_1\beta_2\left(1-e^{-\lambda}\right)^2 exp\left[-\left(1-e^{-\lambda}\right)\left\{\beta_1 x+\beta_2 y\right\}\right]. \quad (4)$$

Substituting equation (4) in equation (3), the reliability coefficient is:

$$r=\beta_1\beta_2\left(1-e^{-\lambda}\right)^2\int_0^{\infty}\int_0^{y}\exp\left[-\left(1-e^{-\lambda}\right)\left\{\beta_1 x+\beta_2 y\right\}\right]dxdy.$$

After simplification the reliability coefficient turned out to be:

$$r=\frac{\beta_1}{\beta_1+\beta_2} \quad (5)$$

The maximum likelihood estimate of the reliability coefficient given in (5) is:

$$\hat{r}=\frac{\hat{\beta}_1}{\hat{\beta}_1+\hat{\beta}_2} \quad (6)$$

In equation (6) $\hat{\beta}_1$ and $\hat{\beta}_2$ are the maximum likelihood estimates of β_1 and β_2 based on samples of sizes "*n*" and "*m*" respectively. The maximum likelihood estimates $\hat{\beta}_1$ and $\hat{\beta}_2$ from distributions (2) and (3) is:

$$\hat{\beta}_1=\frac{n}{\left(1-e^{-\lambda}\right)\sum_{i=1}^{n}x_i}\;;\;\hat{\beta}_2=\frac{m}{\left(1-e^{-\lambda}\right)\sum_{i=1}^{m}y_i}, \quad (7)$$

Using the estimates from (7) we have derived the exact sampling distribution of (6) in the following section.

Sampling Distribution of Reliability Coefficient

In this section we have derived the exact sampling distribution of (6). To find the exact sampling distribution of (6) we first find the distribution of $\sum_{i=1}^{n}x_i$ and $\sum_{i=1}^{m}y_i$. Since *X* has distribution (1), which is Exponential with parameter $\left\{\beta_1\left(1-e^{-\lambda}\right)\right\}^{-1}$, so the distribution of $\sum_{i=1}^{n}x_i$ is $G\left[n;\left\{\beta_1\left(1-e^{-\lambda}\right)\right\}^{-1}\right]$. Similarly, the distribution of $\sum_{i=1}^{m}y_i$

is $G\left[m\,;\left\{\beta_2\left(1-e^{-\lambda}\right)\right\}^{-1}\right]$. Further, the distribution of

$w_1 = n\left\{\left(1-e^{-\lambda}\right)\sum_{i=1}^{n} x_i\right\}^{-1}$ has the density function:

$$f\left(w_1\right)=\left(\frac{n}{\beta_1}\right)^n \frac{1}{\Gamma\left(n\right)\left(1-e^{-\lambda}\right)^{2n}} w_1^{-(n+1)} exp\left[-\frac{n}{\beta_1 w_1\left(1-e^{-\lambda}\right)^2}\right];\ w_1>0\,.$$

Similarly the distribution of $w_2 = m\left\{\left(1-e^{-\lambda}\right)\sum_{i=1}^{m} y_i\right\}^{-1}$ is:

$$f\left(w_2\right)=\left(\frac{m}{\beta_2}\right)^m \frac{1}{\Gamma\left(m\right)\left(1-e^{-\lambda}\right)^{2m}} w_2^{-(m+1)} exp\left[-\frac{m}{\beta_2 w_2\left(1-e^{-\lambda}\right)^2}\right];\ w_2>0\,.$$

Since w_1 and w_2 are independent, so there joint distribution is:

$$f\left(w_1,w_2\right)=\left(\frac{n}{\beta_1}\right)^n\left(\frac{m}{\beta_2}\right)^m \frac{1}{\Gamma\left(n\right)\Gamma\left(m\right)\left(1-e^{-\lambda}\right)^{2(n+m)}} w_1^{-(n+1)} w_2^{-(m+1)}$$

$$exp\left[-\frac{1}{\left(1-e^{-\lambda}\right)^2}\left\{\frac{n}{\beta_1 w_1}+\frac{m}{\beta_2 w_2}\right\}\right]$$

(8)

Making the following transformation in (8):

$$r=\frac{w_1}{w_1+w_2} \quad \text{and} \quad v=w_1+w_2\,.$$

The Joint distribution of r and v is:

$$f\left(r,v\right)=\left(\frac{n}{\beta_1}\right)^n\left(\frac{m}{\beta_2}\right)^m \frac{1}{\Gamma\left(n\right)\Gamma\left(m\right)\left(1-e^{-\lambda}\right)^{2(n+m)}} r^{-(n+1)} v^{-(n+m+1)}\left(1-r\right)^{-(m+1)}$$

$$\cdot exp\left[-\frac{1}{v\left(1-e^{-\lambda}\right)^2}\left\{\frac{n}{\beta_1 r}+\frac{m}{\beta_2\left(1-r\right)}\right\}\right];0\le r\le 1,\ v>0\,.$$

(9)

The marginal distribution of r is:

$$f\left(r\right)=\left(\frac{n}{\beta_1}\right)^n\left(\frac{m}{\beta_2}\right)^m \frac{1}{\Gamma\left(n\right)\Gamma\left(m\right)\left(1-e^{-\lambda}\right)^{2(n+m)}} \int_0^\infty r^{-(n+1)} v^{-(n+m+1)}\left(1-r\right)^{-(m+1)}$$

$$exp\left[-\frac{1}{v\left(1-e^{-\lambda}\right)^2}\left\{\frac{n}{\beta_1 r}+\frac{m}{\beta_2\left(1-r\right)}\right\}\right]dv.$$

(10)

After simplification (10), the marginal distribution of r turned out to be:

$$f\left(r\right)=\frac{\left(m\beta_1\right)^m\left(n\beta_2\right)^n}{\beta\left(m,n\right)} r^{m-1}\left(1-r\right)^{n-1}\left[\left\{m\beta_1 r+n\beta_2\left(1-r\right)\right\}\right]^{-(m+n)};$$

$$0\le r\le 1.$$

(11)

This distribution is a slight modification of Beta Type-I distribution.
In the following section we have obtained the moments of the distribution.

Moments of the Distribution of Reliability Coefficient

The exact sampling distribution of the reliability coefficient is given in (11). Further we have obtained the moments of the reliability coefficient given in (6). Now, the *k*–th moment of the distribution (11) is defined as:

$$E\left(R^k\right)=\int_0^1 r^k f\left(r\right)dr$$

Using the density (11), we get

$$E\left(R^K\right)=\frac{\left(m\beta_1\right)^m\left(n\beta_2\right)^n}{\beta\left(m,n\right)}\int_0^1 r^{k+m-1}\left(1-r\right)^{n-1}\left[\left\{m\beta_1 r+n\beta_2\left(1-r\right)\right\}\right]^{-(m+n)}dr.$$

After simplification, the K-th moment of the reliability coefficient is obtained as:

$$E\left(R^k\right)=\frac{\left(m\beta_1\right)^m}{\left(n\beta_2\right)^n}\frac{\beta\left(m+k,n\right)}{\beta\left(m,n\right)}\,{}_2F_1\left(m+k,m+n,m+n+k;1-\frac{m\beta_1}{n\beta_2}\right) \qquad (12)$$

Using k = 1 and k = 2, the mean and second moment of reliability coefficient turned out to be:

$$E\left(R\right)=\frac{\left(m\beta_1\right)^m}{\left(n\beta_2\right)^n}\cdot\frac{m}{m+n}\,{}_2F_1\left(m+1,m+n,m+n+1;1-\frac{m\beta_1}{n\beta_2}\right) \qquad (13)$$

$$E\left(R^2\right)=\frac{\left(m\beta_1\right)^m}{\left(n\beta_2\right)^n}\cdot\frac{m\left(m+1\right)}{\left(m+n\right)\left(m+n+1\right)}\,{}_2F_1\left(m+2,m+n,m+n+2;1-\frac{m\beta_1}{n\beta_2}\right) \qquad (14)$$

Using (13) and (14), the variance can be easily found. Further, if $m = n = t$ and $\beta_1=\beta_2$, then the mean and variance of R are given as:

$$E\left(R\right)=\frac{1}{2} \quad \text{and} \quad Var\left(R\right)=\frac{1}{4\left(2t+1\right)} \qquad (15)$$

The expected value and variance of reliability can be computed for different values of m, n, β_1 and β_2. The effect of m and n is not much significant on expected value of the reliability as compared with β_1 and β_2. We have given the expected value and variance of reliability for different values of β_1 and β_2 whereas the values of m and n is kept at the same level. The following table contain the expected reliability for different values of β_1 and β_2.

Table I
Expected Reliability

β_1 \ β_2	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	0.5000	0.4510	0.3879	0.3362	0.2955	0.2631	0.2369	0.2154	0.1974	0.1821
0.2		0.5000	0.4825	0.4510	0.4183	0.3879	0.3605	0.3362	0.3147	0.2955
0.3			0.5000	0.4911	0.4726	0.4510	0.4291	0.4079	0.3879	0.3693
0.4				0.5000	0.4946	0.4825	0.4674	0.4510	0.4345	0.4183
0.5					0.5000	0.4964	0.4879	0.4767	0.4642	0.4510
0.6						0.5000	0.4974	0.4911	0.4825	0.4726
0.7							0.5000	0.4981	0.4932	0.4864
0.8								0.5000	0.4985	0.4946
0.9									0.5000	0.4988
1.0										0.5000

The variance in reliability is given in the following table.

Table II
Variance in Reliability

β_1 \ β_2	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	0.0119	0.2413	0.4056	0.5172	0.5959	0.6537	0.6976	0.7320	0.7596	0.7821
0.2		0.0119	0.1328	0.2413	0.3316	0.4056	0.4666	0.5172	0.5598	0.5959
0.3			0.0119	0.0930	0.1709	0.2413	0.3034	0.3579	0.4056	0.4475
0.4				0.0119	0.0727	0.1328	0.1893	0.2413	0.2886	0.3316
0.5					0.0119	0.0605	0.1091	0.1559	0.2001	0.2413
0.6						0.0119	0.0523	0.0930	0.1328	0.1709
0.7							0.0119	0.0465	0.0815	0.1159
0.8								0.0119	0.0421	0.0727
0.9									0.0119	0.0387
1.0										0.0119

From table I we can see that the expected reliability is exactly 0.5 if $\beta_1 = \beta_2$. Further the expected reliability decreases as β_2 goes away from β_1. The variance in reliability is at a fixed point for $\beta_1 = \beta_2$ and the variance increases as β_2 goes away from β_1. The empirical results in Table I and II supported the results in equation (12)

Concluding Remarks

We derived and estimated the reliability coefficient for Erlang – truncated Exponential model in this paper. Some distributional properties of the reliability coefficients have been given. We believe that the paper will be useful for the researchers in the filed of reliability.

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