A Note on Symmetrized Max-Plus Algebra

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Abstract

In this paper, besides crisply introducing M+ and $\mathbb{S}$max, we consider a truncated $\mathbb{S}$max (without transitivity), call it minimal $\mathbb{S}$max or ≈ $\mathbb{S}$max, and discuss its application.

Introduction

Over the last few decades several *path algebras* have emerged initially in response to resolving certain issues that arise in weighted graph theory, for example, computing the shortest or the longest path in a graph. During the first phase of its development, several path algebras, following desultory approaches, were developed. However, as research evolved, it was found that a unified theory was underlying all path algebras and idempotent semiring (also called *dioid*) proved to be the unifying thread. The best known example is the *max-plus semiring* in which addition is defined as max (a, b) and *multiplication* as a + b, the latter being distributive over the former.

For instance, Dioid algebra happens to be the right tool to handle synchronization in a linear manner, whereas this phenomenon seems to be nonlinear or even nonsmooth that cannot be treated with conventional algebraic tools. A motivating observation that certain problems of discrete optimization could be linearised over suitable idempotent semirings played a seminal role.

Besides its initial application, essentially found in the study of discrete event systems, the theory of dioids, involving max (or min) and + as the basic operations, is found appropriate for a large class of systems dealing with *input-output* relation, suitably interpreted as an inf. (or a sup.) convolution. Yet another situation where dioid algebra shows up is the asymptotic behaviour of exponential functions. The conventional operations + and × over positive numbers, say, are transferred into max and + respectively by the mapping $x \to lim_{s\to+\infty}$ exp (sx), which is found relevant, for example, in the study of large deviations. Note that certain classes of dioids have been extensively studied under other names (see [Gun 96], for details).

The *max-plus* (also denoted by (max, +) or M^+) algebra is a subclasses of path algebras called *tropical calculus or minima algebra*. The basic advantage of using M^+ lies in getting linearlized a problem which is nonlinear in a conventional algebra system. M^+ has been found useful in diverse fields, for example, graph theory, discrete event systems, transportation network, parallel computations, project management, machine scheduling and language theory (automata with multiplicities) to name a few (see [Cun 79], [BCOQ 92], [Gun 96], [But 08], and others for various details).

Symmetrized max-plus, also denoted by $\mathbb{S}_{max}$ or $(\mathbb{S}, \oplus, \otimes)$ was developed [Gau 92] in response to removing the deficiency of M^+ which was found lacking additive inverses. However, $\mathbb{S}_{max}$ was found lacking one of the characteristics property of the conventional algebra viz; $a - a = 0$. Gaubert [Gau 92] introduced the notion of *balances* to overcome the aforesaid deficiency. But, the balance relation turns out to be non-transitive. In other to achieve transitivity of the balance relation, a new relation, closely related to the balance relation, was introduced.

The max-plus Algebra

This is also called the *tropical calculus or the minima algebr*a. It has evolved basically as a useful tool to evaluate complicated expressions involving diverse mathematical objects, including multiset like objects, and the operations $\cup$, $\cap$ and +. It seems to stem from control theory (see [Cun 79], [Cun 91] and others for details).

Basically, two new operations (on rational numbers), called tropical addition $\oplus$ and tropical multiplication $\otimes$ are introduced to replace $\cup$, $\cap$ and +. In fact, for simplicity, the familiar symbols + for addition and $\times$ or mere juxtaposition for multiplication, used to manipulate rational expressions traditionally, can be retained if there is no confusion. However, in such cases, an equation is flagged with the symbol (T) at the end to indicate its tropical sense of use. As we shall use $\oplus$ and $\otimes$, no use of (T) will be made unless it becomes useful for clarity.

Some key definitions are as follows:

Let x, y, z be rational numbers.

$x \oplus y = z \Leftrightarrow x \cup y = z \Leftrightarrow \max(x, y) = z.$

[or, $x + y = z\ (T) \Leftrightarrow x \cup y = z \Leftrightarrow \max(x, y) = z$]. Note that $\cup$ is replaced by $\oplus$

$x \otimes y = z \Leftrightarrow x + y = z$ [or, $xy = z\ (T) \Leftrightarrow x + y = z$]

In general, max and + are replaced by $\oplus$ and $\otimes$ respectively for easy handling of complicated expressions. Note that both the operations are commutative and associative which enable us to relate these novel algebraic structures to standard linear algebra, for example, the theory of dioids.

$y = x^n\ (T) \Leftrightarrow y = nx$

That is, powers correspond to ordinary multiples.

For any integer n, $x^n = x \otimes x \otimes \ldots \otimes x = x + x + \cdots + x = nx$

For example, $2^3 = 2 \otimes 2 \otimes 2 = 2 + 2 + 2 = 6 = 3 \times 2 = 6$

Also, $x^{-1}(T) = (-1)x = -x =$ multiplicative inverse of x in tropical sense.

For example, $2^{-1} \otimes 4 = -2 + 4 = 2$

Hence, it is easy to see that tropical multiplication turns rational numbers into a commutative group with multiplicative identity 0 and inverse operation: $x \rightarrow -x$

The notion of tropical inverse allows us to define the operation of division as follows:

$\frac{x}{y} = xy^{-1}\ (T) = x + y^{-1}\ (T) \Leftrightarrow x - y$

It follows that $\frac{x}{y} = z(T) \Leftrightarrow x - y = z$

Tropical addition $\oplus$ is idempotent: $x \oplus x = x(T)$ [since $x \oplus x = \max(x, x) = x$]

This feature of $\oplus$ is typical of its diverse applications which, for example, is equivalent to 'least upper bound' operation in lattices, providing connections with lattice ordered semigroup theory.

The distributive law, connecting $\oplus$ and $\otimes$, holds in tropical calculus:

$x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$

This feature of $\otimes$ reflects its significance as a novel operator and not simply as the conventional addition.

Finally, we define $\cap$ tropically as follows:

Since x $\cap$ y = $x + y - (x \cup y)$,

We have $z = x \cap y \Leftrightarrow \quad z = \frac{x \otimes y}{x \oplus y}$

We note that a number of complicated rational expressions and equations can be translated in a much simpler way into tropical calculus.

(see [Wil 03], for example).

Remarks:

R_1. Note that the tropical addition does not admit inverses. In other word, the operation of subtraction is not definable tropically which implies, in turn, that the familiar cancellation law with respect to addition does not hold, that is, $a \oplus \ c = b \oplus \ c \nRightarrow a = b$.

Also, $0 \oplus x \ = x$ iff $x \ \geq 0$,

While $0 \oplus x = 0$ iff $x \leq 0$,

As a result, we cannot obtain a Boolean algebra and related results which require complementation operation.

R_2. As to treating max as the addition denoted by $\oplus$, and the conventional $+$ as 'multiplication' denoted by $\otimes$, it seems to be promoted by the nature of the class of problems to be solved by the (max, +) algebra (see, for example, the "train" example in [BCOQ 92]).

R_3. The operations $\oplus$ and $\otimes$ may not necessarily be confined to operate as max and +, respectively.

Definition: Max-plus algebra.

The algebraic system (M, $\oplus, \otimes$), M^+ for short, where M = $\mathbb{R} \cup \{-\infty\}$, $a \oplus \ b =$ *max*(a, b) $a \otimes \ b = a + b$, and $\otimes$ is distributive over $\oplus$, is called the max-plus algebra.

Note that the additive identity θ of M^+ is $-\infty$ i.e, $-\infty \oplus a = max(-\infty, a) = a,$ for all a $\in M^+$; and the multiplicative identity e of M^+ is 0 i.e. $0 \otimes a = 0 + a = a.$

Also, $a \leq b$ iff $a \oplus b = b.$

Symmetrization of M^+

It is straight forward to observe that M^+ is a *zerosum free idempotent semiring*. Moreover, it is a semifield since all nonzero elements do have a multiplicative inverse. However, it is a bit disappointing that M^+ lacks an additive inverse and as a result, even some simple linear equations do not have a solution. For example, $x \oplus 2 = 1,$ has no solution.

In order to remove this deficiency, a standard technique is exploited i.e. embedding M^+ into a larger system which will have an additive inverse, akin to embedding the system of natural numbers $\aleph$ into that of integers $\mathbb{Z}$. This technique is known as "symmetrization". That is, "$a \oplus x = \theta$ (additive identity), $a \in M^+$, has a solution" is to say that "a is symmetrized" (see [Gau 92], [Pop 00] and others for details).

We outline here the "symmetrized max algebra", denoted by $S_{max,}$ developed in [Gau 92] as an extension of m^+.

Let $\beta = M^+ \times M^+$, and $x = (x_1, x_2)$, $y = (y_1, y_2)$ be elements of β

We define:

1. $x \oplus y = (x_1, x_2) \oplus (y_1, y_2) = (x_1 \oplus x_2, y_1 \oplus y_2).$

2. $x \otimes y = (x_1, x_2) \otimes (y_1, y_2) = ((x_1 \otimes y_1) \oplus (x_2 \otimes y_2), (x_1 \otimes y_2) \oplus (x_2 \otimes y_1)).$

Thus, (θ, θ) is the additive identity and (e, θ) is the unit element (multiplicative identity) of β. Note that the use of the same symbols $\oplus$ and $\otimes$, both for M^+ and S_{max} (where they operate on the elements of β) will be contextually understood.

We define:

$\ominus x = (x_2, x_1)$

$|x| = x_1 \oplus x_2 = max\ (x_1, x_2)$, called the absolute value of x

$x^* = x \ominus x =(|x|, |x|)$, called the "balance" operator

Note that $x \ominus x = x \oplus (\ominus x) = (x_1, x_2) \oplus (x_2, x_1) \equiv (x_1 \oplus x_2, x_2 \oplus x_1) = (|x|, |x|)$.

The following hold:

1. $(\beta, \oplus)$ is associative, commutative and idempotent with additive identity (θ, θ)

2. $(\beta, \otimes)$ is associative, commutative and distributive over $\oplus$ with multiplicative identity (e, θ).

3. The additive identity is absorbing under $\otimes$.

4. The algebraic structure $(\beta, \oplus, \otimes)$ is a commutative dioid.

Descriptively, this algebra is also called the "algebra of pairs".

The following hold:

$u^* = (\ominus u)^* = (u^*)^*$

$u \otimes v^* (u \otimes v)^*$

$\ominus (\ominus u) = u$

$\ominus (u \oplus v) = (\ominus u) \oplus (\ominus v)$

$\ominus (u \otimes v) = (\ominus u) \otimes v = u \otimes (\ominus v)$

Remark:

In the conventional algebra, we have $a - a = 0$, but in the algebra of pairs, $a \ominus a = a^* \neq (\theta, \theta)$ for all $a \in \beta$. This suggests for introducing a new relation on β.

Definition: Balance relation

Let $x = (x_1, x_2)$, $y = (y_1, y_2)$. We say that x balances y, denoted $x \Delta y$, iff $x_1 \oplus y_2 = x_2 \oplus y_1$.

Follows that $a \ominus a = a^* = (|a|, |a|) \Delta (\theta, \theta)$ for all $a \in \beta$.

Some immediate algebraic features of balances are as follows:

$a \vartriangle a$

$a \vartriangle b \Leftrightarrow b \vartriangle a$

$a \vartriangle b \Leftrightarrow a \ominus b \vartriangle \theta$

$a \vartriangle b, c \vartriangle b \Rightarrow a \oplus c \vartriangle b \oplus d$

$a \vartriangle b \Rightarrow ac \vartriangle bc$

Note, however, that $\vartriangle$ is not transitive.

For example, (3, 2) $\vartriangle$(3, 3) $\wedge$ (3, 3) $\vartriangle$(2, 3) $\nRightarrow$ (3, 2) Δ (2, 3).

Accordingly, Δ is not an equivalence relation to give a partition and, by implication, a quotient set.

A new relation R (say), closely related to the balance operator Δ, is defined which is transitive;

$$x\mathrm{R}y = \begin{cases} x\Delta y, & if\ x_1 \neq x_2\ and\ y_1 \neq y_2 \\ x = y, & otherwise \end{cases}$$

Here = is conventional: $x = y \Leftrightarrow (x_1, x_2) = (y_1, y_2) \Leftrightarrow x_1 = x_2, y_1 = y_2$.

Hence, we can define the quotient $\mathbb{S} = \beta/R$

Definition: $\mathbb{S}_{max}$

The algebraic structure $\mathbb{S}_{max}$ = ($\mathbb{S}$, $\oplus$, $\otimes$) is called the symmetrized max-plus algebra

Notes: Besides developing some extra properties of balances, Gaubert [Gau 92] demonstrates that the balance operation can be extended to vectors and matrices by applying it component- wise. Since our interest lies in translating some main results of M^+ algebras into multiset environment, we will not go into many other finer details. Nevertheless, we purpose to develop at this juncture a new algebra and call it a "minimal $\mathbb{S}_{max}$" algebra, abbreviated as $\approx \mathbb{S}_{max}$.

An outline of $\approx \mathbb{S}_{max}$:

We confine our attention to those properties of balances which are only reflexive and symmetric but not transitive. Such a relation is called a *compatibility* relation. We construct an algebra defined by compatibility relation defined on $M^+ \times M^+$,

Definition: Covering of a set

Let X be a given set and let $A = \{A_1, A_2, \ . \ . \ . \ , A_m\}$ where each A_k, $k = 1, 2, \ . \ . \ . \ , m$ is a nonempty subset of X and $\bigcup_{k=1}^{m} A_k = X$, then A is called a cover of X. Note that A_k's are not necessarily disjoint, and hence it may not define a partition.

Definition: Compatibility relation

A relation R in set X is said to be a “compatibility” relation, sometimes denoted by $\approx$, if it is reflexive and symmetric. Obviously, all equivalence relations are compatibility relations. We shall, however, be concerned herewith those compatibility relations which are not equivalence relations.

For an illustration, let us consider $X = \{\{2, 1, 6, 6\}, \{2, 4, 3,\}, \{3, 7, 5\}, \{6, 4, 8\}, \{4, 5, 5\}\} = \{x_1, \ x_2, \ x_3, \ x_4, \ x_5\}$, where $x_1 = \{2, 1, 6, 6\}$, etc. respectively; and R be given by

$R = \{(x, y) \ / \ x, \ \ y \in X \wedge xRy$ if x and y contain some common elements$\}$

Clearly, $\{2, 1, 6, 6\} \ R \ \{2, 4, 3\} \wedge \{2, 4, 3\} \ R \ \{4, 5, 5\} \nRightarrow \{2, 1, 6, 6\} \ R \ \{4, 5, 5\}$

Then R is a compatibility relation if xRy. and x, y are called compatible if xRy. Note that the elements of X could be multisets as well.

Definition

Let X be a set and $\approx$ a compatibility relation on X. A subset $A \subseteq X$ is called a maximal compatibility block if any element of A is compatible to every other elements of A and no other element of $X - A$ is compatible to all other elements of A.

Schematically, the maximal compatibility blocks for a given compatibility relation R can also be viewed as a complete polygon in the graph of R. Thus a triangle is always a complete polygon and for a quadrilateral to be a complete polygon, we need its two diagonals. Also, any element of the set that is related to itself is a maximal compatibility block. Similarly, any two elements which are compatible to one another but to no other elements form a maximal compatibility block.

For example, in the figure 1 below, in this figure {1,2}, {2,3}, {3,4}, {1,4},{5}, and {1,3,4} are maximal compatibility blocks, but {1,2,3,4} is not. The set of all compatibility classes is called *maximal* compatibility class for a given relation.

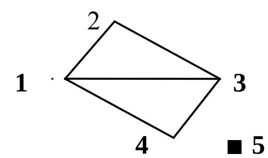

Figure 1

Graph of *R*:

Since ≈ is reflexive and symmetric, in order to draw the graph, it is not necessary to draw the loops at each element nor is it necessary to draw both the edges xRy and yRx. The following is a simplified graph of the compatibility relation *R* described in the aforesaid example:

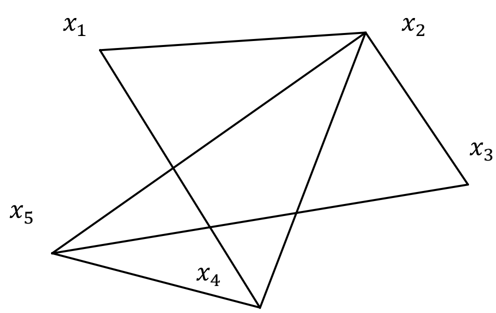

Figure 2

Clearly $A_1 = \{x_1, x_2, x_4\}$, $A_2 = \{x_2, x_3, x_5\}$ and $A_3 = \{x_2, x_4, x_5\}$ are the maximal compatibility blocks corresponding to R on X. Also, A_1 and A_2 form a covering of X. Note that these sets representing maximal compatibility blocks are not disjoint.

Definition:

Compatibility relation on $\mathbf{M}^+ \times \mathbf{M}^+$

Let $x = (x_1,\ x_2),\ y = (y_1, y_2)$. We say that $x \approx$ y iff $x \vartriangle y$.

Hence, it follows that all the properties of $\mathbb{S}max$ algebra, except only those which require transitivity, hold in $\approx \mathbb{S}max$, called "minimal $\mathbb{S}max$".

Conclusion

The prime motivation for introducing $\approx \mathbb{S}max$ lies in the fact that a compatibility relation defines a covering, which has been found useful in solving certain "minimization" problems of switching circuit algebra, particularly for the class of minimization problems that are incompletely specified (see ([TM 97] pp. 171 – 175, 238, and 582), and others for applications).

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References

Baccelli, F. L., Cohen G., Olsder G. J., Quadrat J. P., Synchronization and Linearity: An Algebra for Discrete Event Systems, John Wiley and sons, 1992.

Butkovic P., Introduction to max-algebra, The University of Birminghame (UK) November, 2008.

Cuninghame-Green R.A., Minimax Algebra. *Lecture Notes in Economic and Math. Systems 166. Springer, Berlin, 1979.*

Cuninghame-Green R.A., Minimax Algebra and applications, *Fuzzy Sets and Systems, Vol. 41, No.3, pp. 251-267, 1991.*

Gaubert S. Théorie des Systémes Linéaires dans les dioides, Ph. D. Thesis, *L' Ècole Nationale Supérieure des Mines de Paris, 1992*.

Gunawardena J., An introduction to Idempotency, Basic Research Institute in the Mathematical Sciences HP Laboratories, Bristol HPL-BRIMS – 96 – 24, 1996.

Poplin, P. L., The Semiring of Multisets, Ph. D. Theses, *North Carolina State University, 2000.*

Tremblay J. P. and Manohar R., Discrete Mathematical Structures with Applications to Computer Science, Tata McGraw – Hill Edition, 1997.

Wildberger N.J., A new look at multiset. *School of Mathematics, UNSW Sydney 2052, Australia, 2003.*