Generalized Surface Area of Revolution

Richard Winton, Ph.D. †
Michael L. Warren ‡

Abstract

Suppose a curve C in the plane $\mathbf{R}^2$ is defined by a continuous function over a closed bounded interval. A formula is developed for the radius of revolution from a nonvertical linear axis of revolution L to C. An alternate derivation is also provided. The radius of revolution is then used to produce a formula for the surface area generated by revolving C about L. This result is combined with the standard formula for surface area about a vertical axis to yield a generalized formula for the surface area generated by revolving C about an arbitrary linear axis of revolution.

Introduction

Many applications of derivatives and integrals are routinely studied in calculus. These applications can sometimes be extended to a more general setting than is normally found in the calculus textbooks, such as the result on centroids in [9]. Another concept commonly studied in calculus is that of the surface area generated by revolving a continuous curve about a line in the plane $\mathbf{R}^2$. Some textbooks limit this subject to the revolution of curves about the x and y axes ([1],[2],[5],[6],[7],[8]). In these cases, if a curve C is defined by y = f(x), a ≤ x ≤ b, then the surface area generated is

$$SA = 2\pi\int_a^b |f(x)|\,ds = 2\pi\int_a^b |f(x)|\sqrt{1+[f'(x)]^2}\,dx \qquad (1)$$

when C is revolved about the x-axis and

$$SA = 2\pi\int_a^b |x|\,ds = 2\pi\int_a^b |x|\sqrt{1+[f'(x)]^2}\,dx \qquad (2)$$

when C is revolved about the y-axis, where $ds = \sqrt{1+[f'(x)]^2}\,dx$ is the differential arclength.

Other textbooks, however, include the somewhat more general cases of revolving curves about arbitrary horizontal and vertical lines in $\mathbf{R}^2$ ([3],[4]). In these more general cases, the surface area is given by

$$SA = 2\pi\int_a^b |f(x)-t|\,ds = 2\pi\int_a^b |f(x)-t|\sqrt{1+[f'(x)]^2}\,dx \qquad (3)$$

when C is revolved about the horizontal line y = t and

$$SA = 2\pi\int_a^b |x-t|\,ds = 2\pi\int_a^b |x-t|\sqrt{1+[f'(x)]^2}\,dx \qquad (4)$$

when C is revolved about the vertical line x = t.

The goal of this paper is to develop a formula for the surface area produced by revolving a continuous curve about a completely arbitrary line in $\mathbf{R}^2$. Since vertical lines are not functions, then the surface area produced by revolving a curve about a vertical axis of revolution provided in (4) above must be considered separately. Thus the specific goal here is to develop a formula for the surface area generated by revolving a continuous curve about an arbitrary nonvertical line, greatly generalizing (3) above. The resulting formula, together with (4), will then provide the result sought.

Radius of Revolution

In cases (1) and (3) above, the radius of revolution r of a point P relative to a horizontal axis of revolution L is the vertical distance between P and L. In a similar manner, in cases (2) and (4) above, r is the horizontal distance between the point P and the vertical axis of revolution L. In all of the above cases, r can be described as the length of the unique line segment T in $\mathbf{R}^2$ with the following properties:

(a) One endpoint of T is P.
(b) The other endpoint of T lies on L.
(c) T is perpendicular to L.

Using this general description for r, all four of the above cases can be condensed into the single formula

$$SA = 2\pi\int_a^b r\,ds = 2\pi\int_a^b r\sqrt{1+[f'(x)]^2}\,dx \qquad (5)$$

The goal of this paper then reduces to determining a more general formula for r for all nonvertical axes of revolution, which includes the formula in (3) relative to horizontal lines as a special case. To this end, suppose a curve C is defined by a continuous function y = f(x) for $a \le x \le b$. Suppose further that the axis of revolution L is defined by the linear function A(x) = mx + t, where m and t are real numbers and $m \neq 0$. (See Figure 1.)

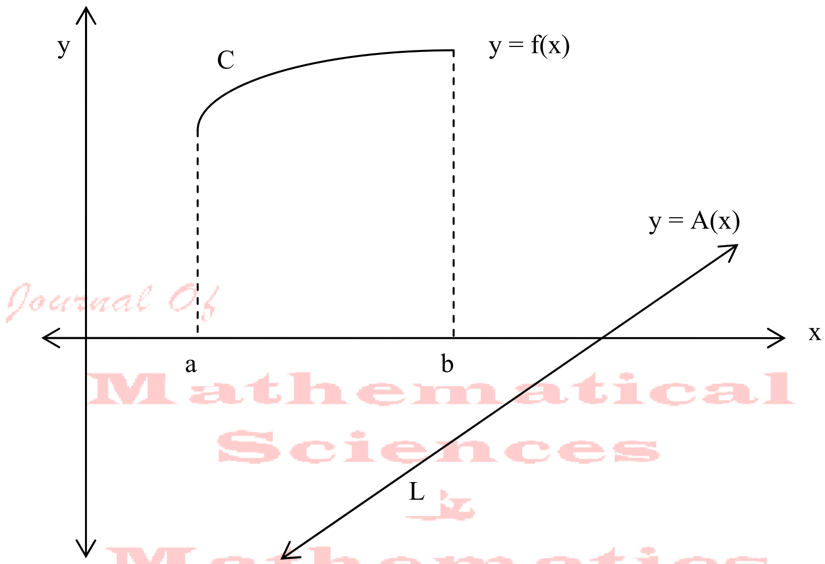

Figure 1

If a ≤ p ≤ b, then the point on C corresponding to x = p is P(p,f(p)). (See Figure 2.)

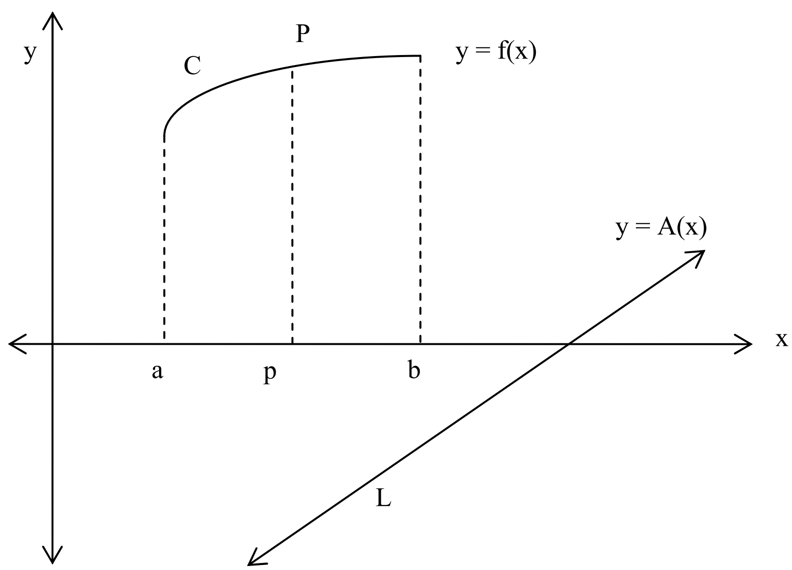

Figure 2

The slope of the line L′ through P and perpendicular to L is $-\frac{1}{m}$. Thus the equation of L′ is $y - f(p) = -\frac{1}{m}(x-p)$, or $y = f(p) - \frac{1}{m}(x-p)$. (See Figure 3.)

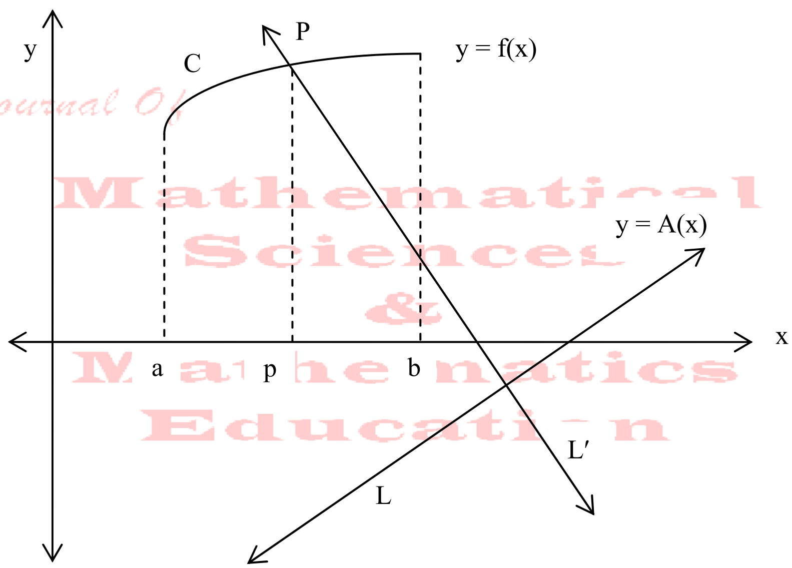

Figure 3

To determine the point of intersection Q of L′ with L, we set mx + t = $f(p) - \frac{1}{m}(x-p)$. Therefore mx + t = $f(p) - \frac{x}{m} + \frac{p}{m}$, so that $mx + \frac{x}{m}$ = $f(p) + \frac{p}{m} - t$. Thus $m^2x + x$ = mf(p) + p − mt, and so x = $\frac{mf(p)+p-mt}{m^2+1}$.

Substituting this expression for x into y = mx + t yields

$$y = m \cdot \frac{mf(p)+p-mt}{m^2+1} + t = \frac{m^2f(p)+mp-m^2t}{m^2+1} + \frac{t(m^2+1)}{m^2+1} =$$

$$\frac{m^2f(p)+mp-m^2t+m^2t+t}{m^2+1} = \frac{m^2f(p)+mp+t}{m^2+1}$$. Thus Q has coordinates

$$Q\left(\frac{mf(p)+p-mt}{m^2+1}, \frac{m^2f(p)+mp+t}{m^2+1}\right)$$. (See Figure 4.)

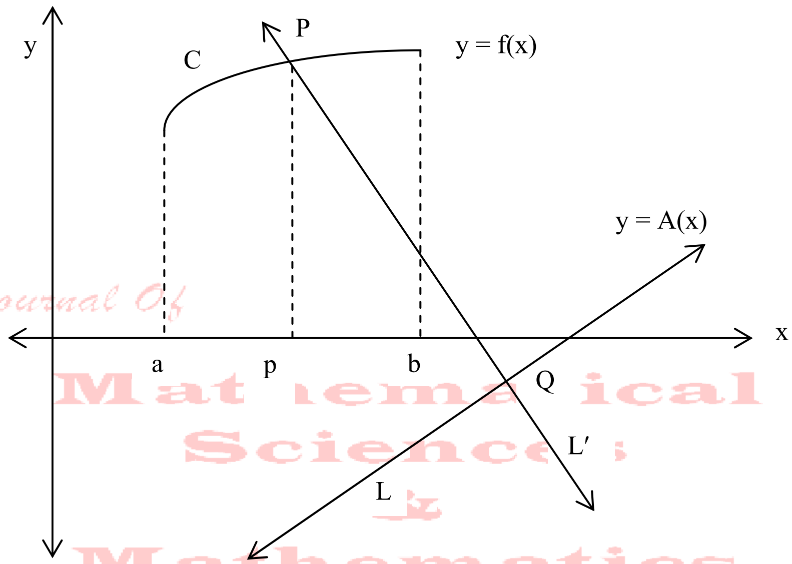

Figure 4

The radius of revolution r of the point P about the axis L is therefore the length of the segment T with endpoints P and Q. Using the distance formula in $\mathbf{R}^2$, we have

$$r = d(P,Q) =$$

$$\sqrt{\left(\frac{mf(p)+p-mt}{m^2+1}-p\right)^2+\left(\frac{m^2f(p)+mp+t}{m^2+1}-f(p)\right)^2} =$$

$$\sqrt{\left(\frac{mf(p)+p-mt-p(m^2+1)}{m^2+1}\right)^2+\left(\frac{m^2f(p)+mp+t-f(p)(m^2+1)}{m^2+1}\right)^2} =$$

$$\frac{\sqrt{\left(mf(p)+p-mt-m^2p-p\right)^2+\left(m^2f(p)+mp+t-m^2f(p)-f(p)\right)^2}}{m^2+1} =$$

$$\frac{\sqrt{\left(mf(p)-m^2p-mt\right)^2+\left(mp+t-f(p)\right)^2}}{m^2+1} =$$

$$\frac{\sqrt{m^2\big(f(p)-mp-t\big)^2+(-1)^2\big(f(p)-mp-t\big)^2}}{m^2+1} =$$

$$\frac{\sqrt{\big(f(p)-mp-t\big)^2(m^2+1)}}{m^2+1} = \frac{\sqrt{\big(f(p)-mp-t\big)^2}\sqrt{m^2+1}}{m^2+1} =$$

$$\frac{\left|f(p)-(mp+t)\right|}{\sqrt{m^2+1}} =$$

$$\frac{\left|f(p)-A(p)\right|}{\sqrt{m^2+1}}. \qquad (6)$$

Alternate Derivation of the Radius of Revolution

A different, and perhaps somewhat less intuitive, derivation of (6) is found in [7, pp. 596-597]. For this alternate approach, suppose θ is the acute angle between the x-axis and the axis of revolution L. Then $\theta = \left|\tan^{-1}(m)\right|$ and $\tan(\theta) = |m|$, so that $\cos(\theta) = \frac{1}{\sqrt{m^2+1}}$. (See Figure 5.)

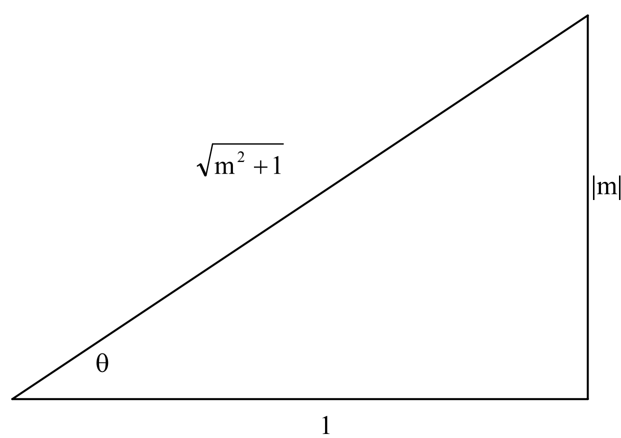

Figure 5

If R is the point on L vertically above or below the point P(p,f(p)), then R has coordinates R(p,A(p)). Thus in triangle PQR we have d(P,R) = $\left|f(p)-A(p)\right|$. Furthermore, ∠QPR is congruent to θ since L′ is perpendicular to L. (See Figure 6.)

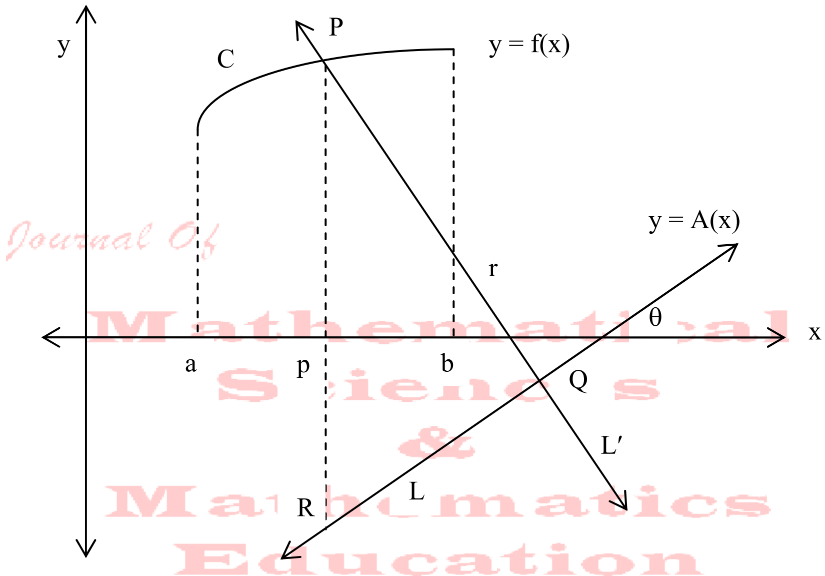

Figure 6

Hence $\frac{r}{|f(p)-A(p)|}$ = cos(θ), and so $r = |f(p)-A(p)| \cdot \cos(\theta) =$

$|f(p)-A(p)| \cdot \frac{1}{\sqrt{m^2+1}} = \frac{|f(p)-A(p)|}{\sqrt{m^2+1}}$, which is consistent with (6).

Surface Area

We are now prepared to generalize formula (3) to include all nonvertical, nonhorizontal axes of revolution. Applying (6) to each value of x for $a \le x \le b$, the radius of revolution of the point (x,f(x)) on the curve C about the axis L is

$$r(x) = \frac{|f(x)-A(x)|}{\sqrt{m^2+1}}.$$

Hence the surface area generated by revolving C about L is

$$SA = 2\pi\int_a^b r(x)ds = 2\pi\int_a^b \frac{|f(x)-A(x)|}{\sqrt{m^2+1}}\sqrt{1+[f'(x)]^2}\,dx =$$

$$\frac{2\pi}{\sqrt{m^2+1}}\int_a^b \left|f(x)-A(x)\right|\sqrt{1+\left[f'(x)\right]^2}\,dx\,, \qquad (7)$$

where $\left|f(x)-A(x)\right|$ is the vertical distance between C and L for each x such that $a \le x \le b$.

Note, however, that when the axis of revolution L is horizontal, then $m = 0$. In this case the equation of L simplifies to $A(x) = t$. Consequently, (7) reduces to $SA = 2\pi\int_a^b \left|f(x)-t\right|\sqrt{1+\left[f'(x)\right]^2}\,dx$, which is consistent with (3) above. Hence the case for horizontal axes of revolution when $m = 0$ is included in (7).

Conclusion

Combining (4) with (7), we have the following conclusion which includes all linear axes of revolution in $\mathbf{R}^2$. If a curve C is defined by a continuous function $y = f(x)$ for $a \le x \le b$, then the surface area generated by revolving C about a linear axis of revolution L is

$$SA = \begin{cases} 2\pi\int_a^b \left|x-t\right|\sqrt{1+\left[f'(x)\right]^2}\,dx & \text{if L is vertical defined by } x = t \\ \dfrac{2\pi}{\sqrt{m^2+1}}\int_a^b \left|f(x)-A(x)\right|\sqrt{1+\left[f'(x)\right]^2}\,dx & \text{if L is defined by } A(x) = mx + t. \end{cases}$$

† *Richard Winton*, *Ph.D.*, Tarleton State University, Texas, USA
‡ *Michael L. Warren*, Tarleton State University, Texas, USA

References

[1] Howard Anton, Irl Bivens, and Stephen Davis, "Calculus", 7th ed., *John Wiley & Sons, Inc.*, New York, 2002.

[2] Ross L. Finney and George B. Thomas, *Addison-Wesley*, New York, 1990.

[3] John B. Fraleigh, "Calculus with Analytic Geometry", 2nd ed., *Addison-Wesley*, Reading, Massachusetts, 1985.

[4] Roland E. Larson, Robert P. Hostetler, and Bruce H. Edwards, "Calculus with Analytic Geometry", 4th ed., *D. C. Heath and Company*, Lexington, Massachusetts, 1990.

[5] Abe Mizrahi and Michael Sullivan, 3rd ed., *Wadsworth*, Belmont, California, 1990.

[6] S. L. Salas and Einar Hille, "Calculus: One and Several Variables", 6th ed., *John Wiley & Sons, Inc.*, New York, 1990.

[7] James Stewart, "Calculus", 5th ed., *Thomson*, United States, 2003.

[8] Earl W. Swokowski, Michael Olinick, and Dennis Pence, "Calculus of a Single Variable", 2nd ed., *PWS Publishing*, Boston, 1994.
[9] Richard Winton, *Symmetry at Infinity*, College Mathematics Journal **36** (2005) 228-231.