A Note On Covering and Partition of A Finite Set

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Abstract

In this short communication, an extremal combinatorial problem concerning partition and covering of a finite set is discussed. It is pointed out that unlike the case with partition, no closed formula solution for determining the total number of coverings is known. Some motivating steps are indicated. The application of compatibility relation to solve some minimization problem is outlined.

Introduction

It may be recalled that counting the number of elements in a set is a broad mathematical problem and a large number of them yet remaining unsolved or partially answered.

However, in a typical combinatorial problem, as the one proposed in this paper, a relatively crisp description and some further structures are provided. Accordingly, such a problem may or may not have outright a closed formula solution even if it appears structurally feasible. In fact, many such problems do exist whereby either no solutions or partial solutions are available. For example, assume that a certain number *n* of objects is given. Is it possible to assign them sets so that each object is in atleast one set, each pair of objects are in exactly one set together, every two sets have exactly one object in common, and no set contains all or all but one of the objects? The answer depends on *n* and is only partially known to this day [7].

In this paper we propose to discuss an extremal combinatorial problem concerning *partition* and *covering* of a finite set.

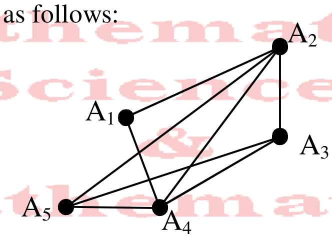
Definition: Covering and partition of a finite set

Let *S* be a nonempty finite set. A decomposition of *S* of the form $S = A_1 \cup A_2 \cup, \cdots, \cup A_k$, where $A_i \neq \phi$ for all $i = 1,2,\cdots,k$ is called a *covering* of *S*. If, in addition, $A_i \cap A_j = \phi$ for $i \neq j$, then $\{A_1, A_2, \cdots, A_k\}$ is called a partition of *S* and the sets $A_1, A_2, \cdots, A_k$ are called the *blocks* of the partition.

It follows that {S} is both a covering and a partition on S.

Note also that an n-set can have atmost n blocks. It is well-known that a relation on S which is reflexive, symmetric and transitive, called an *equivalence relation*, gives a partition of S. A related notion is that of compatibility relation. A relation on S which is reflexive and symmetric is called a *compatibility* relation, sometimes denoted by $\approx$.

Also, let R be a compatible relation on a set X, then $x, y \in X$ are called compatible if $x \; R \; y$.
For example,
Let $X = \{A_1, A_2, A_3, A_4, A_5\}$ where $A_1 = \{1,2\}$, $A_2 = \{2,3,4\}$,
$A_3 = \{3,4,5\}$, $A_4 = \{2,4\}$
$A_5 = \{4,5\}$ and let R be given by
$R = \{(A_i, A_j) / A_i, A_j \in X \wedge A_i R A_j$ if A_i and A_j contain some common element$\}$.
Clearly, $A_1 \, R \, A_2 \wedge A_2 \, R \, A_3$, but $\neg(A_1 R A_3)$.
A simplified diagram is as follows:



Note that, in the case of a compatibility relation, it is not necessary to draw loops at each element nor is it necessary to draw both xRy and yRx.
Since a compatibility relation is not necessarily transitive, it does not necessarily define a partition; however, it does define a covering. A number of pleasing properties of compatibility relation have been investigated (see [4] and [5]) and used in solving certain minimization prolems of switching theory, especially when the specification provided is incomplete.
From the diagram above, it is easy to observe that the elements in each of the sets $\{A_1, A_2, A_4\}$, $\{A_2, A_3, A_4\}$, $\{A_2, A_4, A_5\}$, $\{A_3, A_4, A_5\}$ and $\{A_2, A_3, A_4, A_5\}$ are mutually compatible and the sets are not mutually disjoint. Accordingly, it does define only a covering of X. Infact, there may exist more than one combination of these sets to yield the covering of X.
Further, in order to solve minimization problems, the following elaboration is in order.
A subset $A \subseteq X$ is said to be a maximal compatibility block if any element of A is compatible to all other elements of A and no element of $X - A$ is compatible to all the elements of A. Moreover, any element of the set which is related only to itself, and any two elements compatible to one another but to no other elements of the set are also the maximal compatibility blocks.
It follows that a maximal compatibility block is the largest complete polygon in which every vertex is related to its every other vertex. For example, in the diagram above, $\{A_1, A_2, A_3, A_4\}$ is not a complete polygon.

Clearly, $\{A_1, A_2, A_4\}$ and $\{A_2, A_3, A_4, A_5\}$ are the only maximal compatibility blocks. Note that no other subset of *X* forms a compatibility block (see [4] and [5] for more details).

Problem

Let S be a non-empty set with n elements, usually called an *n-set*. Find: *(a)* the size of the family of all the partitions of S, denoted by $|par(s)|$, and *(b)* the size of the family of all the coverings of S, denoted by $|cov(s)|$.
Note that a relation between two partitions or two coverings or a partition and a covering will be understood by treating them as sets.

Results

Fact 1. $|cov(s)| > |par(s)|, for\ |s| \geq 2$. Proof follows by definition.
Fact 2. In order to answer *(a),* one would wish to determine the number of all *possible* distinct equivalence relations on S, for which no general method is known to exist. Nevertheless, in view of the fact that there exists a one to one correspondence between the classes of distinct equivalence relations defined on a set and the partition of that set, a combinatorial formulation is known as follows (see [1], [6], and many others):
The number of partitions of an n-set into k blocks is given by $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ or $S(n,k)$, called a Starling number of the *second* kind. Follows, for $n \geq 1$,

$$\left\{ \begin{matrix} n \\ 1 \end{matrix} \right\} = 1, \qquad \left\{ \begin{matrix} n \\ 2 \end{matrix} \right\} = 2^{n-1} - 1,$$

$$\left\{ \begin{matrix} n \\ n-1 \end{matrix} \right\} = \binom{n}{2}, \qquad \left\{ \begin{matrix} n \\ n \end{matrix} \right\} = 1 \cdot$$

Also for $n \geq k \geq 1$,

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\} + k \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\}$$

(see [1], p.18 and p.46 for the proof).
Note that this recurrence relation has a precedent in the theory of integer partition viz.,
for $n \geq k \geq 2$,

$$P_k(n) = P_{k-1}(n-1) + P_k(n-k)$$

(see [1], p.14 for the proof).
Moreover, in view of the existence of a correspondence between set partitions and surjective mappings, the following is an explicit formula for stirling number of the second kind viz.,

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \frac{1}{k\,!} \sum_{j=1}^{k} (-1)^{k-j} (k_j)^{j^n}$$

(see [1], p.19 and p.46 for the proof).

The total number of partitions of an n-set, given by $\sum_{k=1}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}$,

is called a Bell number denoted by B(n). Also, B(n+1) = $\sum_{k=0}^{n} \binom{n}{k} \bullet$ B(k).
Note, however, that no explicit formulation is known for p(n), the number of partitions of the integer n ([1], p.14).
In order to compute p(n) of all partitions of any n-set, the following nice

formula is constructed in [3]:

$$\frac{p(n)}{n\,!}$$ = nth coefficient of the analytic function

f(x) = exp(exp(n)-1) in its standard representation in the form of Taylor series.

Fact 3. No general formulation is known to explicitly determine $|cov(s)|$.

We putforward some motivating steps in this regard.

Step 1. It is known that the total number of nonempty subsets of an n-set S = 2n -1.

Step 2. As noted above, the total number of partitions of an n-set S=B(n), the Bell number = the sets of all possible combinations of nonempty disjoint subsets of S which form a partition of S.

Step 3. Now, if one knows the number of all possible combinations of subsets of S which are not disjoint and form a covering of S, then the sum of the two numbers determined in step 2 and step 3 would answer (b).

However, to our knowledge, no closed formula for this is known.
Nevertheless, the following is a further known reflection in this regard. The largest number (say ℓ) of subsets of an n-set S one can have, if no two of them are disjoint, is half the number of all subsets of S; that is, ℓ = (2n / 2) = 2n-1. The proof follows from the fact that between any subset T and its complement S - T, atmost one can be chosen. In order to obtain the aforesaid half of the number, one element x of S is chosen and all the subsets of S that contain x are tracked. Clearly, the family of all such subsets of S would form a covering of S, for each element x of S. Accordingly, the number (say C) of all such coverings would be n.
For example, if S = {1,2,3}, then ℓ= 4, and {{1},{1,2},{1,3},{1,2,3}} is a covering of S corresponding to the choice of the element 1 of S, and C = 4.

Note, however, that besides C = 4, there are many other coverings of S; for example, {{1}, {2},{1,2},{1,3},{1,2,3}}, etc.

At this end, we mention without proof a very nice result obtained in [2] proving a best possible bound on the number of subsets of a given finite set that form its covering.

Let n,s,t be integers with $s > t > 1$ and $n > (t+2) 2s-t-1$. If n subsets of a set S with s elements have union S then some t of them have union S. The result is best possible ([2], p197).

Concluding Remarks

In view of the fact that a relation which is reflexive and symmetric (but not necessarily transitive) has been found characteristically useful in explicating a variety of problems e.g., minimization problems of switching theory (see [4] and [5] for some details); it has increasingly become pertinent to closely investigate various issues in this regard.

In the first place, besides providing a sufficiently rich partition calculus, the paper puts forward some challenging problems of varying degrees specially from the pedagogical point of view; for example, constructing an algorithm to compute a maximal or minimal number of subsets of an n-set under varying conditions that would cover the set. In turn, for a given compatibility relation on an n-set, it is a good exercise to compute maximal compatibility blocks and coverings.

The paper also delineates an 'open' combinatorial problem to formulate a closed formula to determine the size of the family of all the coverings of an n-set and provides some motivating steps to solve it.

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