Boundary Points of Subcontinua in Homogeneous Metric Continua

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Abstract

A brief history of the problem contained in this paper is provided. Basic definitions are also provided due to some variation in the literature. Preliminary results related to separation in homogeneous continua are developed to provide access to the main theorem. The existence of a homeomorphism with special properties on homogeneous metric continua is established. It is determined that in one class of homogeneous metric continua a natural partition exists for the set of boundary points of subcontinua with minimal boundary. Finally, concluding remarks offer some possibilities for extending the results of this paper.

Introduction

In 1980 Simmons [11] developed several results on separation in homogeneous continua. These results were extended in 2007 and 2009 by Winton ([14],[15]). Earlier in 1975, Hagopian [6] established the existence of a homeomorphism on homogeneous metric continua which maps an arbitrary point to an arbitrary image within a neighborhood without mapping any point in the space beyond a prescribed distance from its preimage. These results will be extended and applied to produce a very natural partition for the set of boundary points of subcontinua with minimal three-point boundaries. Due to some variation throughout the literature, we begin with basic definitions and notations.

Definitions

In the most general sense, a continuum is a compact, connected, Hausdorff topological space. A metric continuum is a compact, connected metric space. If (X,ρ) is a metric space, $x \in X$, and r is a positive real number, then the sphere about x of radius r is $S(x,r) = \{y \in X \mid \rho(x,y) < r\}$.

If H is a subset of a topological space X, then Int(H), Cl(H), and Bd(H) are the topological interior, closure, and boundary of H, respectively. A separation $A \mid B$ of H is a partition of H into nonempty relatively open sets A and B. Furthermore, H separates X if and only if X is connected but X–H is not connected.

If X is a topological space and n is an integer greater than 1, then an n-pod of X is a subcontinuum of X whose boundary contains precisely n points. Furthermore, n is the pod number of X if and only if X contains an n-pod but X contains no k-pod whenever k is an integer and $1 < k < n$. The pod number of X is denoted by P(X).

Preliminary Results

In 1980 Simmons showed that if X is a homogeneous continuum with pod number 2, H is a 2-pod in X, $x \in \mathrm{Int}(H)$, $y \in X-H$, and $\{x,y\}$ separates X, then $\{x\}$ separates H [11, Lemma 4]. However, this result cannot be extended for larger values of $P(X)$. For in 2009 Winton showed that if X is a homogeneous continuum with pod number $n \geq 3$ and H is an n-pod in X, then no single point can separate H [15, Theorem 6]. A direct application of this result appears in the following lemma.

Lemma 1: Suppose X is a homogeneous continuum, $P(X) = 3$, H is a 3-pod in X, $x \in \mathrm{Int}(H)$, and $y,z \in X-H$. Then $\{x,y,z\}$ does not separate X.

Proof: If $\{x,y,z\}$ separates X, then there is a separation $A|B$ of $X-\{x,y,z\}$. Define $U = A \cap H$ and $V = B \cap H$. Since $\{x,y,z\}$ is closed in X ([7, p. 64, Corollary 3.12],[12, p. 130, Theorem A]), then $X-\{x,y,z\}$ is open in X. Furthermore, since A and B are open in $X-\{x,y,z\}$ and $X-\{x,y,z\}$ is open in X, then A and B are open in X [15, Lemma 1(c)]. Therefore U and V are open in H. However, $A \subseteq X-\{x\}$ and $B \subseteq X-\{x\}$, so that $U \subseteq H-\{x\} \subseteq H$ and $V \subseteq H-\{x\} \subseteq H$. Thus U and V are open in $H-\{x\}$ [15, Lemma 1(a)].

Assume $U = A \cap H = \varnothing$, so that $A \subseteq X-H$. Since $A|B$ is a separation of $X-\{x,y,z\}$, then A is open in $X-\{x,y,z\}$. Furthermore, since $\{x\}$ is closed in X ([7, p. 64, Corollary 3.12],[12, p. 130, Theorem A]), then $X-\{x\}$ is open in X. Therefore $X-\{x,y,z\} = (X-\{y,z\}) \cap (X-\{x\})$ is open in $X-\{y,z\}$. Thus A is open in $X-\{x,y,z\}$ and $X-\{x,y,z\}$ is open in $X-\{y,z\}$, so that A is open in $X-\{y,z\}$ [15, Lemma 1(c)]. Similarly B is open in $X-\{y,z\}$.

Define $D = B \cup \{x\}$, so that $D \neq \varnothing$ and $D \subseteq B \cup \mathrm{Int}(H)$ since $x \in \mathrm{Int}(H)$. Furthermore, since $A \cap H = \varnothing$ then $H \subseteq X-A$. Since $y,z \in X-H$ then $H-\{x\} = H-\{x,y,z\} = H \cap (X-\{x,y,z\}) \subseteq (X-A) \cap (X-\{x,y,z\}) = B$ since $A|B$ is a separation of $X-\{x,y,z\}$. Therefore $H = (H-\{x\}) \cup \{x\} \subseteq B \cup \{x\} = D$, and so $B \cup \mathrm{Int}(H) \subseteq B \cup H \subseteq B \cup D = D$. Hence $D = B \cup \mathrm{Int}(H)$. However, $\mathrm{Int}(H) \subseteq X-\{y,z\}$ since $y,z \in X-H$. Since $\mathrm{Int}(H)$ is open in X and $\mathrm{Int}(H) \subseteq X-\{y,z\} \subseteq X$, then $\mathrm{Int}(H)$ is open in $X-\{y,z\}$ [15, Lemma 1(a)]. Since B and $\mathrm{Int}(H)$ are open in $X-\{y,z\}$, then $D = B \cup \mathrm{Int}(H)$ is open in $X-\{y,z\}$.

Thus A and D are nonempty open sets in $X-\{y,z\}$. Furthermore, $A \cap D = A \cap (B \cup \{x\}) = (A \cap B) \cup (A \cap \{x\}) = \varnothing$ since $A|B$ is a separation of $X-\{x,y,z\}$, $x \in \mathrm{Int}(H)$, and $A \cap H = \varnothing$ (by assumption). Finally, $A \cup D = A \cup (B \cup \{x\}) = (A \cup B) \cup \{x\} = (X-\{x,y,z\}) \cup \{x\} = X-\{y,z\}$. Hence $A|D$ is a separation of $X-\{y,z\}$, and so $A \cup \{y,z\}$ and $D \cup \{y,z\}$ are 2-pods in X with boundary $\{y,z\}$ ([11, Lemma 1],[14, Lemma 2]). However, this is a contradiction since $P(X) = 3$, and so $U \neq \varnothing$. Similarly $V \neq \varnothing$.

Thus U and V are nonempty open sets in $H-\{x\}$. Furthermore, $U \cap V \subseteq A \cap B = \varnothing$, and $U \cup V = (A \cap H) \cup (B \cap H) = (A \cup B) \cap H = (X-\{x,y,z\}) \cap H =$

H–{x,y,z} = H–{x} since y,z∈X–H. Thus U$|$V is a separation of H–{x}, and so {x} separates H. This is a contradiction since P(X) = 3 and H is a 3-pod in X [15, Theorem 6]. Therefore {x,y,z} does not separate X.

The result of Lemma 1 can be applied to the complementary 3-pod of H based on the work of Winton [14]. The symmetrical result produced in Lemma 2 will make the proofs of Lemma 4 and Theorem 5 more efficient.

Lemma 2: Suppose X is a homogeneous continuum, P(X) = 3, H is a 3-pod in X, x∈X–H, and y,z∈Int(H). Then {x,y,z} does not separate X.

Proof: Since H is a 3-pod in X, then K = Cl(X–H) is also a 3-pod in X with Bd(K) = Bd(H) [14, Theorem 6]. Furthermore, since H is compact and X is Hausdorff, then H is closed in X ([1, p. 81, Corollary 5.13],[3, p. 165, Theorem 6.4]). Thus Bd(H) ⊆ H ([3, p. 105, Theorem 4.5(6)],[9, p. 90, Theorem 1(vi)],[12, p. 69, (3)]), and so (X–H)∩Bd(H) = ∅. Therefore X–H =
[(X–H)∪Bd(H)]–Bd(H) = [(X–H)∪Bd(X–H)]–Bd(K) = Cl(X–H)–Bd(K)
([10, p. 87, no. 12],[13, p. 28, Theorem 3.14(a)]) = K–Bd(K) = Int(K)
([8, p. 46, Theorem 10],[13, p. 28, Theorem 3.14(b)]). Similarly X–K = Int(H).

Alternatively, {Int(H),Bd(H),X–H} and {Int(K),Bd(K),X–K} are partitions of X ([2, p. 142, Theorem 30.2],[4, p. 72, Theorem 4.11(4)]). Furthermore, since K = Cl(X–H), then Bd(K) = Bd(H) [14, Theorem 6], Int(H)∩Int(K) = ∅, and (X–H)∩(X–K) = ∅. Therefore X–H = Int(K) and X–K = Int(H).

Thus K is a 3-pod in X, x∈X–H = Int(K), and y,z∈Int(H) = X–K. Hence {x,y,z} does not separate X by Lemma 1.

We will now apply a result of Hagopian [6] to establish the existence of a homeomorphism with special separation properties on any homogeneous metric continuum X. This homeomorphism will allow each point in a finite subset of X to remain relatively close to its preimage while mapping any point x in X to any other point in a neighborhood of x.

Theorem 3: Suppose that (X,ρ) is a homogeneous metric continuum, x∈X, $\{x_i\}_{i=1}^{n} \subseteq X$, and N_i is a neighborhood of x_i for $1 \le i \le n$. Then there exists a neighborhood N of x with the property that for each y∈N, there is a homeomorphism h:X→X such that h(x) = y and $h(x_i) \in N_i$ for $1 \le i \le n$.

Proof: Since N_i is a neighborhood of x_i, then for each i, $1 \le i \le n$, there exists a real number $\varepsilon_i > 0$ such that $x_i \in S(x_i, \varepsilon_i) \subseteq N_i$. Define $\varepsilon = \min\{\varepsilon_i\}_{i=1}^{n}$. Hence there exists a neighborhood N of x with the property that for each y,z∈N, there is a homeomorphism h:X→X such that h(y) = z and $\rho(v, h(v)) < \varepsilon$ for each v∈X [6, Lemma 4]. Consequently, for each y∈N, there is a homeomorphism $h:X\to X$ such that $h(x) = y$ and $\rho(x_i, h(x_i)) < \varepsilon \le \varepsilon_i$ for $1 \le i \le n$, and so $h(x_i) \in S(x_i, \varepsilon_i) \subseteq N_i$. Therefore $h(x) = y$ and $h(x_i) \in N_i$ for $1 \le i \le n$.

In homogeneous metric continua with pod number 2, it is possible for 2-pods to have exactly one common boundary point. For example, suppose X is the unit circle in the Cartesian plane. Then the arc from the point (0,1) clockwise to (1,0) and the arc from (1,0) clockwise to (0,–1) are 2-pods in X with exactly one common boundary point (1,0). Furthermore, any two distinct arcs with exactly one common endpoint constitute a pair of 2-pods containing exactly one common boundary point. However, this property fails in homogeneous metric continua with pod number 3. More specifically, it will be shown that in homogeneous metric continua with pod number 3, the boundaries of 3-pods are either disjoint or coincide.

Lemma 4: Suppose X is a homogeneous metric continuum, $P(X) = 3$, and H and K are 3-pods in X. Then either $Bd(H)\cap Bd(K) = \varnothing$ or $Bd(H) = Bd(K)$.

Proof: Define $n = |Bd(H)\cap Bd(K)|$. Since H and K are 3-pods in X, then $n\in\{0,1,2,3\}$. Suppose $Bd(H) = \{x,y,z\}$ and $Bd(K) = \{p,q,r\}$. Therefore $Int(K)\,|\,(X-K)$ is a separation of $X-\{p,q,r\}$ [14, Lemma 3].

Suppose $n = 1$. Without loss of generality, assume $x = p$, so that $y,z\notin\{p,q,r\}$. Therefore $y,z\in X-\{p,q,r\} = Int(K)\cup(X-K)$.

Case 1: If $y,z\in Int(K)$, then $Int(K)$ is a neighborhood of both y and z. By Theorem 3 (or the Weak Effros Property ([5],[11])) there exists a neighborhood N of x with the property that for each $u\in N$, there is a homeomorphism $h:X\to X$ such that $h(x) = u$ and $h(y),h(z)\in Int(K)$. Since $x = p\in Bd(K) = Bd(X-K)$ and N is a neighborhood of x, then $N\cap(X-K) \ne \varnothing$, and so there exists some $x'\in N\cap(X-K)$. Since $x'\in N$ then there is a homeomorphism $h:X\to X$ such that $h(x) = x'$ and $h(y),h(z)\in Int(K)$. Define $y' = h(y)$ and $z' = h(z)$. Since $\{x,y,z\} = Bd(H)$ then $\{x,y,z\}$ separates X [14, Lemma 3]. Furthermore, since h is a homeomorphism, then $\{x',y',z'\} = \{h(x),h(y),h(z)\}$ also separates X. However, this contradicts Lemma 2 since $x'\in N\cap(X-K)\subseteq X-K$ and $y',z'\in Int(K)$.

Case 2: On the other hand, if $y,z\in X-K$, then $y,z\in Int(M)$, where $M = Cl(X-K)$ is the complementary 3-pod of K in X [14, Theorem 6]. By an argument similar to that above in Case 1, there is a homeomorphism $h:X\to X$ such that $x' = h(x)\in Int(K) = X-M$, $\{y',z'\} = \{h(y),h(z)\} \subseteq X-K = Int(M)$, and $\{x',y',z'\}$ separates X. This again contradicts Lemma 2. (Equivalently, use the fact that $x' = h(x)\in Int(K)$, $\{y',z'\} \subseteq X-K$, and $\{x',y',z'\}$ separates X to contradict Lemma 1.)

Case 3: Finally, if $y\in Int(K)$ and $z\in X-K$, then an argument similar to that above in Case 1 will guarantee the existence of a homeomorphism $h:X\to X$ such that $x' = h(x)\in X-K$, $y' = h(y)\in Int(K)$, $z' = h(z)\in X-K$, and $\{x',y',z'\}$ separates

X. If $y \in X-K$ and $z \in \mathrm{Int}(K)$, then reverse the roles of y and z in the above argument to produce a homeomorphism $h:X \to X$ such that $x' = h(x) \in X-K$, $y' = h(y) \in X-K$, $z' = h(z) \in \mathrm{Int}(K)$, and $\{x',y',z'\}$ separates X. In either case, Lemma 1 is contradicted.

Thus all possible cases for $n = 1$ contradict either Lemma 1 or Lemma 2. Hence $n = 1$ is impossible.

Now suppose that $n = 2$. Without loss of generality, assume $x = p$ and $y = q$, so that $z \notin \{p,q,r\}$. Since it was argued above that $\mathrm{Int}(K)\,|\,(X-K)$ is a separation of $X-\{p,q,r\}$, then $z \in X-\{p,q,r\} = \mathrm{Int}(K) \cup (X-K)$.

Case 4: Suppose $z \in \mathrm{Int}(K)$. Since $y = q \in \mathrm{Bd}(K)$, then an argument similar to that above in Case 1 produces a homeomorphism $h:X \to X$ such that $y' = h(y) \in X-K$ and $z' = h(z) \in \mathrm{Int}(K)$. Furthermore, since $\mathrm{Int}(K)\,|\,(X-K)$ is a separation of $X-\mathrm{Bd}(K)$, (or by ([2, p. 142, Theorem 30.2],[4, p. 72, Theorem 4.11(4)])), then $\{\mathrm{Int}(K),\mathrm{Bd}(K),X-K\}$ is a partition of X. Thus if $x' = h(x)$, then exactly one of $\mathrm{Int}(K)$, $\mathrm{Bd}(K)$, and $X-K$ contains x'.

Firstly, suppose that $x' \in \mathrm{Bd}(K)$. Since h is a homeomorphism, then h(H) and K are 3-pods in X with $\mathrm{Bd}[h(H)] = \{x',y',z'\}$. Since $y' \in X-K$ and $\{\mathrm{Int}(K),\mathrm{Bd}(K),X-K\}$ is a partition of X, then $y' \notin \mathrm{Bd}(K)$. Furthermore, since $z' \in \mathrm{Int}(K) = K-\mathrm{Bd}(K)$ ([8, p. 46, Theorem 10],[13, p. 28, Theorem 3.14(b)]), then $z' \notin \mathrm{Bd}(K)$. Therefore $\mathrm{Bd}[h(H)] \cap \mathrm{Bd}(K) = \{x'\}$. Thus $|\mathrm{Bd}[h(H)] \cap \mathrm{Bd}(K)| = 1$, which is impossible according to the argument above for $n = 1$.

Secondly, suppose that $x' \in \mathrm{Int}(K)$. Then $x',z' \in \mathrm{Int}(K)$ and $y' \in X-K$. Therefore $\{x',y',z'\}$ does not separate X by Lemma 2. However, this is a contradiction since $\{x',y',z'\} = \mathrm{Bd}[h(H)]$ [14, Lemma 3].

Thirdly, suppose that $x' \in X-K$. Then $z' \in \mathrm{Int}(K)$ and $x',y' \in X-K$. Therefore $\{x',y',z'\}$ does not separate X by Lemma 1. This is again a contradiction since $\{x',y',z'\} = \mathrm{Bd}[h(H)]$ [14, Lemma 3].

Case 5: On the other hand, if $z \in X-K$, then define $M = \mathrm{Cl}(X-K)$. Thus M is a 3-pod with $\mathrm{Bd}(M) = \mathrm{Bd}(K)$ [14, Theorem 6]. Furthermore, since $\mathrm{Bd}(M) = \mathrm{Bd}(K) \subseteq K$ ([3, p. 105, Theorem 4.5(6)],[9, p. 90, Theorem 1(vi)],[12, p. 69, (3)]), then $\mathrm{Int}(M) = M-\mathrm{Bd}(M)$ ([8, p. 46, Theorem 10],[13, p. 28, Theorem 3.14(b)]) $= M-\mathrm{Bd}(K) = X-K$, so that $z \in X-K = \mathrm{Int}(M)$.

Alternatively, $\{\mathrm{Int}(K),\mathrm{Bd}(K),X-K\}$ and $\{\mathrm{Int}(M),\mathrm{Bd}(M),X-M\}$ are partitions of X ([2, p. 142, Theorem 30.2],[4, p. 72, Theorem 4.11(4)]). Furthermore, since $M = \mathrm{Cl}(X-K)$, then $\mathrm{Bd}(M) = \mathrm{Bd}(K)$ [14, Theorem 6], $\mathrm{Int}(K) \cap \mathrm{Int}(M) = \varnothing$, and $(X-K) \cap (X-M) = \varnothing$. Therefore $X-K = \mathrm{Int}(M)$ and $X-M = \mathrm{Int}(K)$, so that $z \in X-K = \mathrm{Int}(M)$.

Since $x,y \in \mathrm{Bd}(K) = \mathrm{Bd}(M)$ and $z \in \mathrm{Int}(M)$, then replacing K with M in the argument above in Case 4 produces the same contradictions. Therefore $n = 2$ is also impossible. Since $n \in \{0,1,2,3\}$, then either $n = 0$ or $n = 3$. Hence either $\mathrm{Bd}(H) \cap \mathrm{Bd}(K) = \varnothing$ or $\mathrm{Bd}(H) = \mathrm{Bd}(K)$.

We are now prepared to present the main result of the paper. Lemma 4 provides a most natural way in which to partition the set of all boundary points of 3-pods in an arbitrary homogeneous metric continuum X with pod number 3. The cells of this partition are actually the boundaries of the 3-pods in X.

Main Theorem

Theorem 5: Suppose X is a homogeneous metric continuum, P(X) = 3, $T = \{H \mid H \text{ is a 3–pod in } X\}$, $S = \{x \in X \mid x \in Bd(H) \text{ for some } H \in T\}$, and $P = \{Bd(H) \mid H \in T\}$. Then P is a partition of S.

Proof: Since P(X) = 3, then X contains a 3-pod $M \in T$. Therefore $Bd(M) \in P$, so that $P \neq \varnothing$. For each $H \in T$, H is a 3-pod. Therefore $|Bd(H)| = 3 > 0$, so that $Bd(H) \neq \varnothing$. Furthermore, it is clear that $Bd(H) \subseteq S$. Thus P is a nonempty collection of nonempty subsets of S. If $H, K \in T$, then H and K are 3-pods in X. Then by Lemma 4, either $Bd(H) \cap Bd(K) = \varnothing$ or $Bd(H) = Bd(K)$. Finally, $x \in S$ if and only if $x \in Bd(M)$ for some $M \in T$ if and only if $x \in \bigcup_{H \in T} Bd(H)$. Therefore $S = \bigcup_{H \in T} Bd(H)$. Hence P is a partition of S.

Concluding Remarks

The comments preceding Lemma 4, along with Lemma 4 and Theorem 5, establish a significant difference between the properties of boundary points of homogeneous metric continua with pod number 2 and those with pod number 3. Other differences between these two classes of homogeneous continua have already been established. For example, homogeneous continua with pod number P(X) = n can be separated by a single point if and only if n = 2 [15, Corollary 7]. It is possible that the results of this paper can be used to establish additional differences and possibly even common properties between these two classes of homogeneous metric continua. Perhaps these results can then be extended for homogeneous metric continua with pod number n > 3.

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References

1. Cain, G. L., *Introduction to General Topology*, Addison Wesley Publishing Company, Reading, Massachusetts, 1994.
2. Christie, D. E., *Basic Topology*, Macmillan Publishing Company, New York, 1976.
3. Croom, F. H., *Principles of Topology*, Saunders College Publishing, Philadelphia, 1989.
4. Dugundji, J., *Topology*, Allyn and Bacon, Inc., Boston, 1966.
5. Effros, E. G. *Transformation Groups and C* Algebras*, Annals of Mathematics, Vol. 81, (1965) pp. 38-55.
6. Hagopian, C. L., *Homogeneous Plane Continua*, Houston Journal of

Mathematics, Vol. 1, (1975) pp. 35-41.
7. Hall, D. and Spencer, G., *Elementary Topology*, John Wiley & Sons, Inc., New York, 1955.
8. Kelley, J. L., *General Topology*, D. Van Nostrand Company, Inc., Princeton, New Jersey, 1955.
9. McCarty, G., *Topology*, Dover Publications, Inc., New York, 1967.
10. Mendelson, B., *Introduction to Topology*, third edition, Dover Publications, Inc., New York, 1990.
11. Simmons, F. W., *When Homogeneous Continua Are Hausdorff Circles (or Yes, We Hausdorff Bananas)*, Continua, Decompositions, and Manifolds, University of Texas Press, (1980) pp. 62-73.
12. Simmons, G. F., *Introduction to Topology and Modern Analysis*, McGraw Hill Book Company, Inc., San Francisco, 1963.
13. Willard, S., *General Topology*, Addison Wesley Publishing Company, Inc., Reading, Massachusetts, 1970.
14. Winton, R., *Complementary Subcontinua*, Journal of Mathematical Sciences and Mathematics Education, Vol. 2, No. 2, (2007) pp. 15-19.
15. Winton, R., *Separation in Homogeneous Continua*, Journal of Mathematical Sciences and Mathematics Education, Vol. 4, No. 1, (2009) pp. 34-41.