A Triangle Embedding Problem

Richard Winton, Ph.D. †

Abstract

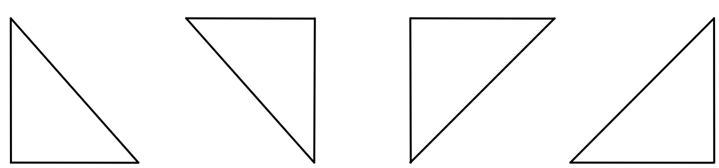
For each ordered pair (m,n) of positive integers, T(m,n) is defined to be the number of triangles with prescribed properties which are contained in a predetermined m×n rectangle. This paper will derive a general closed formula for T(m,n). Several increasingly refined formulas pertaining to an important special case are also developed.

Introduction

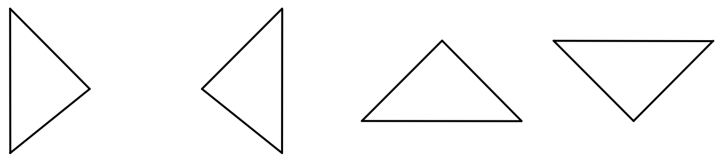
In 1991 Bowman [1] defined T(n), for each positive integer n, to be the number of triangles contained in the subset [0,n]×[0,n] of the Cartesian plane $\mathbf{R}^2$ whose sides are contained in lines which pass through at least one point with integer coordinates with slope 1, −1, 0, or ∞. He then posed the problem of deriving a closed formula for T(n). In 1993 Komanda [2] presented a solution to this problem by showing that T(n) = $\left[\!\left[3n^3+\frac{9}{2}n^2+n\right]\!\right]$, where $[\![x]\!]$ is the greatest integer less than or equal to x for each real number x. In order to generalize the problem presented by Bowman [1], the square subset [0,n]×[0,n] of $\mathbf{R}^2$ is replaced with the rectangular subset [0,m]×[0,n], where m and n are positive integers. To this end, we let **N** denote the set of positive integers. The function T:**N**×**N**→**N** is defined so that T(m,n) is the number of triangles contained in the subset [0,m]×[0,n] of $\mathbf{R}^2$ whose sides are contained in lines which pass through at least one point with integer coordinates with slope 1, −1, 0, or ∞.

General Strategy

The triangles such as those described above can be classified as one of eight distinct types based on their shape and orientation as shown in Figure 1 below.



type 1 type 2 type 3 type 4



type 5 type 6 type 7 type 8

Figure 1

We divide such triangles into three classes containing types 1 through 4, 5 through 6, and 7 through 8. Thus using symmetry, if A(m,n), B(m,n), and C(m,n) are the number of triangles of types 1, 5, and 7, respectively, contained in $[0,m]\times[0,n]$, then

$$T(m,n) = 4A(m,n) + 2B(m,n) + 2C(m,n). \qquad (1)$$

Calculation of A(m,n)

Suppose that $m \geq n$. In order for the line containing the vertical side of a type 1 triangle to contain a point with integer coordinates, the equation of the line must be $x = b$ for some integer b. Hence the first coordinates of the vertices on the vertical side of the triangle must be b. Similarly, the second coordinates of the vertices on the horizontal side of the triangle must be c for some integer c. Thus the coordinates of the lower left vertex are (b,c). Suppose the coordinates of the upper left and lower right vertices of the triangle are (b,v) and (u,c), respectively. Since the slope between (b,v) and (u,c) is -1, the line containing these points has equation $y-v = -(x-b)$, or equivalently $v = x+y-b$. Furthermore, the line must contain some point (p,q) with integer coordinates p and q, and so $v = p+q-b$ is an integer. Similarly u is an integer, and so each vertex of a type 1 triangle has coordinates both of which are integers. Such a vertex will be called an *integer vertex*.

Thus the horizontal and vertical sides of a type 1 triangle have equal integer lengths. If each of these sides has length k, then $1 \leq k \leq \min\{m,n\}$ for the triangle to be contained in $[0,m]\times[0,n]$. Since it was assumed that $m \geq n$ then we have $1 \leq k \leq n$. Furthermore, since a type 1 triangle has integer vertices, then adjacent horizontal and vertical positions differ by a shift of one unit. Consequently such a triangle can occupy any one of $m+1-k$ horizontal positions and $n+1-k$ vertical positions in order to be contained in $[0,m]\times[0,n]$. Therefore the number of type 1 triangles contained in $[0,m]\times[0,n]$ whose legs have integer length k is $(m+1-k)(n+1-k)$.

Hence $A(m,n) = \sum_{k=1}^{n}(m+1-k)(n+1-k) = (m+1-1)(n+1-1) + (m+1-2)(n+1-2)$
$+ \cdots + (m+1-(n-1))(n+1-(n-1)) + (m+1-n)(n+1-n) = mn + (m-1)(n-1)$

$+ \cdots +$ (m+2−n)2 + (m+1−n)1. Reversing both the order of these terms and the order of the factors in each term produces $A(m,n) = \sum_{k=1}^{n} k(m+k-n) =$

$\sum_{k=1}^{n}[k^2+(m-n)k] = \sum_{k=1}^{n} k^2 + (m-n)\sum_{k=1}^{n} k = \frac{1}{6}n(n+1)(2n+1) + (m-n)\frac{1}{2}n(n+1)$

$= \frac{1}{6}n(n+1)(3m-n+1)$.

On the other hand, if $m \le n$ then, in a manner similar to that above, we have $A(m,n) = \frac{1}{6}m(m+1)(3n-m+1)$. Thus in general, if $\alpha = \min\{m,n\}$ and $\beta = \max\{m,n\}$, then

$$A(m,n) = \frac{1}{6}\alpha(\alpha+1)(3\beta-\alpha+1). \qquad (2)$$

Calculation of B(m,n)

Similar to the argument above for type 1 triangles, the vertical side of a type 5 triangle must be contained in a line with equation $x = b$ for some integer b. Hence the first coordinates of the vertices on this side are b. If the second coordinate of the top vertex is v, then the side with slope −1 is contained in the line with equation $y-v = -(x-b)$, or $v = x+y-b$. Furthermore, the line must contain some point (p,q) with integer coordinates p and q, and so $v = p+q-b$ is an integer. Hence the top vertex is an integer vertex. Similarly, the bottom vertex on the vertical side of the triangle is also an integer vertex, and so the vertical side has integer length. Thus if the length of the vertical side is k, where k an integer and $1 \le k \le n$, then the triangle can occupy any one of $n+1-k$ vertical positions in order to satisfy the requirements that it has integer vertices on the vertical side and is contained in $[0,m]\times[0,n]$.

Now suppose such a type 5 triangle with vertical side of length k can occupy λ distinct horizontal positions for some integer λ. Since the vertical side of a type 5 triangle has integer vertices, then adjacent horizontal positions differ by a shift of one unit. Furthermore, the right vertex of the left most triangle in a given vertical position has first coordinate $\frac{k}{2}$, where $\frac{k}{2} \le m$. Therefore the right most triangle for any given vertical position will have a right vertex with first coordinate $\frac{k}{2}+(\lambda-1)$. Since all such triangles must be contained in $[0,m]\times[0,n]$, then $\frac{k}{2}+(\lambda-1) \le m$. Thus $\lambda \le m+1-\frac{k}{2}$, where λ is an integer, and so

$\lambda = \left[\left[m+1-\frac{k}{2} \right]\right]$ is the greatest integer less than or equal to $m+1-\frac{k}{2}$.

Therefore the number of type 5 triangles contained in $[0,m]\times[0,n]$ with vertical side of length k is $\left[\left[m+1-\frac{k}{2}\right]\right](n+1-k)$. Since $1 \le k \le n$ and $\frac{k}{2} \le m$, which implies that $k \le 2m$, then $1 \le k \le \min\{2m,n\}$. Hence if $\rho = \min\{2m,n\}$, then

$$B(m,n) = \sum_{k=1}^{\rho}\left[\left[m+1-\frac{k}{2}\right]\right](n+1-k). \tag{3}$$

Calculation of C(m,n)

Using symmetry and reversing the roles of m and n in the argument for B(m,n) above, a type 7 triangle with horizontal side of integer length k can occupy $m+1-k$ distinct horizontal positions and $\left[\left[n+1-\frac{k}{2}\right]\right]$ distinct vertical positions. Therefore the number of type 7 triangles contained in $[0,m]\times[0,n]$ with horizontal side of length k is $(m+1-k)\left[\left[n+1-\frac{k}{2}\right]\right]$. Furthermore, the length k of the horizontal side must satisfy both $1 \le k \le m$ and $\frac{k}{2} \le n$, which implies that $k \le 2n$. Therefore $1 \le k \le \min\{m,2n\}$. Hence if $\sigma = \min\{m,2n\}$, then

$$C(m,n) = \sum_{k=1}^{\sigma}(m+1-k)\left[\left[n+1-\frac{k}{2}\right]\right]. \tag{4}$$

General Formula for T(m,n)

Having derived formulas for A(m,n), B(m,n), and C(m,n), we are now prepared to construct a general formula for T(m,n). Therefore, substituting (2), (3), and (4) into (1) produces

$$T(m,n) = \frac{2}{3}\alpha(\alpha+1)(3\beta-\alpha+1) + 2\sum_{k=1}^{\rho}\left[\left[m+1-\frac{k}{2}\right]\right](n+1-k) + 2\sum_{k=1}^{\sigma}(m+1-k)\left[\left[n+1-\frac{k}{2}\right]\right], \tag{5}$$

where $\alpha = \min\{m,n\}$, $\beta = \max\{m,n\}$, $\rho = \min\{2m,n\}$, and $\sigma = \min\{m,2n\}$.

Special Case

When m = n, the triangles in question are contained in the square $[0,n]\times[0,n]$ in $\mathbf{R}^2$. In this case we have $\alpha = \min\{n,n\} = n$ and $\beta = \max\{n,n\} = n$. Therefore $A(n,n) = \frac{1}{6}n(n+1)(3n-n+1)$ (by (2) above) $= \frac{1}{6}n(n+1)(2n+1)$. The formulas for the k^{th} terms of B(m,n) and C(m,n) in (3) and (4), respectively, both reduce to $\left[\!\left[n+1-\frac{k}{2} \right]\!\right](n+1-k)$. Furthermore, the upper summation index for B(m,n) in (3) above becomes $\rho = \min\{2n,n\} = n$. Similarly, the upper summation index for C(m,n) in (4) above becomes $\sigma = \min\{n,2n\} = n$. Therefore $B(n,n) = C(n,n) = \sum_{k=1}^{n}\left[\!\left[n+1-\frac{k}{2} \right]\!\right](n+1-k)$. Consequently the formulas for T(m,n) in (1) and (5) above reduce to

$$T(n,n) = 4A(n,n) + 4B(n,n) = \frac{2}{3}n(n+1)(2n+1) + 4\sum_{k=1}^{n}\left[\!\left[n+1-\frac{k}{2} \right]\!\right](n+1-k). \tag{6}$$

Refinement of B(n,n)

Define $b_k = \left[\!\left[n+1-\frac{k}{2} \right]\!\right](n+1-k)$ for each positive integer k, so that $B(n,n) = \sum_{k=1}^{n} b_k$. If k is odd, then $\left[\!\left[n+1-\frac{k}{2} \right]\!\right] = \left(n+1-\frac{k}{2}\right)-\frac{1}{2} = \frac{1}{2}(2n+1-k)$, and so

$$b_k = \frac{1}{2}(2n+1-k)(n+1-k). \tag{7}$$

On the other hand, if k is even, then $\left[\!\left[n+1-\frac{k}{2} \right]\!\right] = n+1-\frac{k}{2} = \frac{1}{2}(2n+2-k)$, and so

$$b_k = \frac{1}{2}(2n+2-k)(n+1-k). \tag{8}$$

Thus for each positive integer k, $b_{2k-1} = \frac{1}{2}(2n+1-(2k-1))(n+1-(2k-1))$ (by (7)) $= 2k^2 - (3n+4)k + (n+1)(n+2)$ and $b_{2k} = \frac{1}{2}(2n+2-2k)(n+1-2k)$ (by (8)) $= 2k^2 - 3(n+1)k + (n+1)^2$. Hence

$$b_{2k-1} + b_{2k} = 4k^2 - (6n+7)k + (n+1)(2n+3). \qquad (9)$$

If n is odd, then $B(n,n) = \sum_{k=1}^{n} b_k = b_n + \sum_{k=1}^{n-1} b_k = b_n + \sum_{k=1}^{(n-1)/2} [b_{2k-1} + b_{2k}]$

$= \frac{1}{2}(2n+1-n)(n+1-n) + \sum_{k=1}^{(n-1)/2} [4k^2 - (6n+7)k + (n+1)(2n+3)]$ (by (7) and (9)) =

$$\frac{1}{24}(n+1)(10n^2 + 5n - 3). \qquad (10)$$

However, if n is even, then $B(n,n) = \sum_{k=1}^{n} b_k = \sum_{k=1}^{n/2} [b_{2k-1} + b_{2k}] =$

$\sum_{k=1}^{n/2} [4k^2 - (6n+7)k + (n+1)(2n+3)]$ (by (9)) =

$$\frac{1}{24} n(10n^2 + 15n + 2). \qquad (11)$$

Refinement of T(n,n)

Thus if n is odd, then substituting (10) into (6) yields T(n,n) =
$4A(n,n) + 4B(n,n) = \frac{2}{3} n(n+1)(2n+1) + 4 \cdot \frac{1}{24}(n+1)(10n^2 + 5n - 3) =$

$$\frac{1}{2}(6n^3 + 9n^2 + 2n - 1). \qquad (12)$$

On the other hand, if n is even, then substituting (11) into (6) yields T(n,n) =
$4A(n,n) + 4B(n,n) = \frac{2}{3} n(n+1)(2n+1) + 4 \cdot \frac{1}{24} n(10n^2 + 15n + 2) =$

$$\frac{1}{2}(6n^3 + 9n^2 + 2n). \qquad (13)$$

Combining (12) and (13), we have

$$T(n,n) = \begin{cases} \frac{1}{2}(6n^3 + 9n^2 + 2n - 1) & \text{if } n \text{ is odd} \\ \frac{1}{2}(6n^3 + 9n^2 + 2n) & \text{if } n \text{ is even} \end{cases}. \qquad (14)$$

Clearly the formula in (14) can then be condensed to

$$T(n,n) = \left[\left[\frac{1}{2}\left(6n^3 + 9n^2 + 2n\right)\right]\right] = \left[\left[3n^3 + \frac{9}{2}n^2 + n\right]\right], \qquad (15)$$

which is consistent with the results of Komanda [2].

† *Richard Winton*, *Ph.D.*, Tarleton State University, Texas, USA

References

1. Bowman, Douglas, *Problem E3450*, The American Mathematical Monthly, Vol. 98, No. 6, (1991) p. 553.
2. Komanda, Nasha, *Problem E3450*, The American Mathematical Monthly, Vol. 100, No. 3, (1993) p. 300.