Using Fuzzy Sets to Determine the Continuity of the van Hiele Levels

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Abstract

An important problem in the van Hiele level theory of geometric thinking is whether the levels are discontinuous or continuous. The purpose of the present study is to resolve this unsettled question. An index of fuzziness is used to measure the fuzziness of student groups in a three-dimensional geometry test. Participants were 50 students in three groups who had the same mathematical background. The results show high fuzziness of the student groups which implies that the levels are not separate of each other and hence continuous. This work provides indirect support for the strong effect of continuity on student difficulty in geometric tasks because the difficulty increases as the fuzziness increases.

Background

The van Hiele level theory of geometric thinking (Hoffer, 1983; van Hiele,1986; Wirszup,1976) describes the ways of student thinking in Euclidean geometry. It emanates from Aristotle's thesis which states that everything that exists is constructed in the form of a taxonomy (Adler, 1978). This theory can be considered a Hegelian phenomenology since it examines the different levels of manifestation of intelligence in the epistemological route towards knowledge. Its cognitive activity is characterized by five hierarchical and qualitatively different levels in the development of student geometric thinking.

A higher level entails the knowledge that was explicit in any lower level and contains some additional knowledge that was implicit at lower levels. This is Johann Herbart's apperception process of assimilating new knowledge to old knowledge which means that the unknown is understood in terms of the Known (Weber, 1960). Each level appears as a metatheory of the previous one because verbalization of the way of learning at one level becomes the subject matter at the next level. This inductive characteristic of levels depends on the appearance of new geometric concepts and definitions and the transitions between levels happen under the influence of language, geometric content and teaching methods.

The following is a synopsis of the van Hiele levels.

Level 1 (Visual). The elements of study are geometric objects. This level is characterized by student perception of geometric figures in their totality as entities according to their appearance.

Level 2 (Analysis). The elements of study are properties that analyze the geometric objects. Students begin to discern the properties of geometric figures, make an analysis of figures perceived and recognize them by their properties.

Level 3 (Abstraction). The elements of study are statements that relate the properties. Students can establish relations among the properties of a geometric figure and see that one property follows from another.
Level 4 (Deduction). The elements of study are sequences of the statements. Students grasp the significance of deduction as a means of constructing and developing geometric theory.
Level 5 (Rigor). The elements of study are properties that analyze the sequences. Students understand the foundations of geometry and the properties of a deductive system.

According to van Hiele (1986), the levels are discontinuous (discrete), that is, there is lack of coherence between the structures (networks of relations) of the levels. This means that the levels are separate of each other and the transition of a level to the next one is not gradual but abrupt. A number of researchers have focused on this van Hiele's claim, but they have not been, successful in confirming it.

Burner and Shaughnessy (1986) noted that the observations of transitional thinking and reviewers' occasional difficulties in deciding between levels while making level assignments may suggest that the levels are "more continuous in nature than their discrete descriptions would lead one to believe." Van Hiele (1986) claims that the corresponding contents of different levels sometimes conflict. Fuys, Geddes and Tischler (1988) state that results of their study are mixed on the van Hiele's claim that the levels are discontinuous. Wilson (1990), in a reanalysis of the CDASSG project (Usiskin, 1982), found an overlap between levels. Gutierrez, Jaime and Fortuny (1991), after surveying the van Hiele-based research, theorized that the levels are not discontinuous and continuity in the levels means that "the acquisition of a specific level does not happen instantaneously or very quickly but rather can take several months or even years." Zadeh (1973) states that "the key elements in human thinking are not numbers, but labels of fuzzy sets, that is, classes of objects in which the transition from membership to non-membership is gradual rather than abrupt."

The information above offers indications suggesting that the levels are continuous, that is, there is coherence between their structures which means that they are not separate of each other. The aim of this study is to verify that the van Hiele levels are continuous using a fuzzy approach.

Methods and Procedures

Several philosophers and fuzzy theorists (Black, 1937; Kemeny, 1959; Zadeh, 1965) have been concerned for some time with the fact that humans think and communicate through fuzzy information. The kinds of categories people use in communication and thought have gradations and are not delimited by sharp boundaries. Fuzziness seems to be the quality of vagueness in making distinctions in the world, some form of indeterminacy of communication and thought or incompleteness in knowledge. It results from the inability to define all words precisely. The meaning of the word "old" is different for most people and the dividing line that people create to make their decision between old and young is therefore a vague one. From this idea of vagueness comes the idea of fuzziness.

The concept of fuzziness opens the way to fuzzy set theory which was introduced in 1965 by L.A.Zadeh to handle the fuzziness arising from the subjectivity that is inherent in thinking and judgment (Klir and Folger, 1988; Smithson,1987; Zadeh,1965). The fundamental concept of fuzzy set theory is the concept of fuzzy set. A fuzzy set is a set with imprecise boundaries, that is, the change from, non-membership to membership in a fuzzy set is gradual rather than abrupt. An element may belong partly to the fuzzy set. Formally, a fuzzy set A in a universal set U is characterized by a membership function m_A: U→ [0,1]. As $m_A(u)$ approaches l, the membership degree of u in A increases. The membership degrees lie somewhere in the real interval [0,1] or may be linguistic terms. The linguistic term "old" is vague and imprecise but it can be defined in the form of fuzzy set. A 65 years old person is considered old by most people but a 85 years old person might not see it that way. A person looking at a crowd may decide who is old and therefore has created a fuzzy set of old people. In this fuzzy set belong people who are "really" old and "fairly" old with different membership degrees. The most important feature of fuzzy set is to describe a fuzzy (ill-defined) event and to express the amount of fuzziness in human thinking and judgment.

An object may be defined as a part of the world that is distinguished as a single entity and characterized by several variables, that is, images of attributes of the object. If a variable has values that may be considered as labels of fuzzy sets it is called fuzzy variable. If a set of fuzzy variables is distinguished on an object, then there is a fuzzy system on the object (Klir & Wierman, 1998). The values of a fuzzy variable may be observed against some sort of background like time, space or population. An observation expressed in terms of a fuzzy variable with n values (fuzzy sets) is a n-tuple of membership degrees, one membership degree for each fuzzy set.

Gutierrez, Jaime end Fortuny (1991) offered a three-dimensional geometry paradigm to evaluate the acquisition of the van Hiele levels. The present work is based on the data of this study which is a creation of human mind and can be considered as an object. On this object one observes the following fuzzy variables: acquisition of Level 1, acquisition of Level 2, acquisition of Level 3 and acquisition of Level 4 that may be denoted by v_1, v_2, v_3, and v_4 respectively. Each variable has the following values: no acquisition, low acquisition, intermediate acquisition, high acquisition and complete acquisition that can represent fuzzy sets and be denoted by a, b, c, d and e respectively. Therefore, a fuzzy system is distinguished on this object. The background that distinguishes different observations of fuzzy variables is the student groups A, B and C with 20, 21 and 9 students respectively. Gutierrez et al. (1991) tabulated the number of students attaining degrees of acquisition of each van Hiele level in Table 1. Notice that the first four levels are considered since Level 5 does not appear in secondary school work.

Each observation consists of four 5-tuples of numbers, one 5-tuple for each fuzzy variable v_1, v_2, v_3 and v_4 and one number for each fuzzy set a, b, c, d and e (Table 2). The numbers represent the membership degrees of the observed value of each fuzzy variable (attribute) in the five fuzzy sets. For student group B, the membership degree of the observed value of fuzzy variable v_2 (acquisition of Level 2) in the fuzzy set c (intermediate acquisition) is 4/21. This means that 4 of the 21 students of group B (see Table 1) had intermediate acquisition of van Hiele Level 2.

The pervasiveness of fuzziness in student group geometric thinking suggests the use of fuzzy sets for its measure. It seems reasonable to consider that if the fuzziness is high in at least one of two consecutive fuzzy variables, then the coherence of the structures of the corresponding levels is high. This means that the levels are not separate of each other and hence continuous. In the following an index is used to calculate the fuzziness of student group geometric thinking and hence to determine whether the levels are continuous.

The fuzziness involved in a fuzzy system can be calculated by the Shannon-Wiener diversity index (Pielou, 1975). This is basically an entropy measure of the form

$$F = -\sum_{i=1}^{n} m_i \ln m_i / \ln \mathrm{n}$$

namely minus the sum over all fuzzy sets n of the product of the partial membership (membership degree) and its natural logarithm. This sum is divided by the natural logarithm of the number of fuzzy sets n in order to normalize this fuzzy measure so that the maximum value is 1 regardless of the number of fuzzy sets.

The fuzziness of the student groups A, B and C in the fuzzy system above is calculated by this index using Table 2 and is shown in Table 3. For example, the fuzziness of student group A working on fuzzy variable v_3 (acquisition of Level 3) with values (fuzzy sets) a, b, c, d and e (standing for no, low, intermediate, high and complete acquisition respectively) is

$$\begin{aligned} F = & -[(2/20)\ln(2/20)+(3/20)\ln(3/20)+(6/20)\ln(6/20)+(6/20)\ln(6/20)+ \\ & (3/20)\ln(3/20)]/\ln 5 \\ & = -[(-0.230)+(-0.285)+(-0.361)+(-0.361)+(-0.285)]/1.609 \\ & = 1.522/1.609 \\ & = 0.946 \end{aligned}$$

where the membership degrees for the fuzzy sets a, b, c, d and e are $m_1 = 2/20$, $m_2 = 3/20$, $m_3 = 6/20$, $m_4 = 6/20$ and $m_5 = 3/20$ respectively.

Results and Conclusions

The results in Table 3 help one to infer the following: Since at least one of the values of fuzziness of a student group that correspond to two consecutive fuzzy variables is high (>0.500), the corresponding levels are not separate of each other and hence continuous. For example, the values of fuzziness of the student group A in fuzzy variables v_1 (acquisition of Level 1) and v_2 (acquisition of Level 2) are 0 and 0.710 respectively. Since 0.710>0.500 the corresponding levels, Level 1 and Level 2, are not separate of each other and hence continuous. Notice that the zero fuzziness of student group C in fuzzy variables v_3 and v_4 means that this student group had no acquisition of Level 3 and of Level 4.

This study gave an exegesis of continuity and confirmed that the van Hiele levels are continuous, that is, are coherent and not separate of each other. Since continuity depends on fuzziness it has a strong effect on students' difficulties in geometric tasks. It seems that continuity is responsible for students' increasing number of difficulties in understanding geometry and the occasional difficulties that reviewers and teachers have in deciding between levels while making level assignments.

The fuzzy system above is a humanistic system on the van Hiele model which possesses the characteristics of organized complexity, that is, it has a moderate number of variables and shows the essential features of organization (Klir and Folger, 1988). Humanistic systems of organized complexity appear also on other models of thinking, learning and development such as SOLO taxonomy (Biggs and Collis,1982). Future research results, that may appear from the application of Shannon-Wiener index on these models, will probably establish this index as a viable measure in educational research.

The methods used in this study may be considered a qualitative innovation in mathematics education research since they can incorporate the judgments of experts and make better use of uncertain and imprecise data. They are well suited for examining student group geometric thinking because it involves holistic descriptions and inherent imprecision of concepts. The fuzzy set theory may be considered a methodology in mathematics education since it has a coherent collection of methods for the acquisition of new knowledge which can be used for the solution of a wide variety of problems.

A case can be made that greater emphasis be given to this methodology since its partnership with empirical work can produce fruitful results. Efforts have been made to embed fuzzy techniques in the van Hiele level theory of geometric thinking. Elements of "soft" mathematics (fuzzy sets, possibilities) have been used to mathematize the van Hiele levels, study the transitions inside each level, rank spatial perception categories, measure the student group capacity for obtaining geometric information and study the transitions across levels (Perdikaris, 1996a,1996b, 1998, 2002,2004). However, the fuzzy set theory, eventhough it can play a role in mathematics education, has not given sufficient attention and has yet to establish a place of its own, as a research tool, in mathematics education research.

Table 1
Number of Students Attaining Degrees of Acquisition of Each van Hiele Level (Adopted from Gutierrez et al.. 1991)

Group	van Hiele Level	Degree of acquisition: No acquisition	Low	Intermediate	High	Complete
A	1	0	0	0	0	20
A	2	1	0	3	6	10
A	3	2	3	6	6	3
A	4	13	7	0	0	0
B	1	0	0	1	2	18
B	2	0	3	4	13	1
B	3	9	6	5	1	0
B	4	16	5	0	0	0
C	1	0	2	4	2	1
C	2	3	4	2	0	0
C	3	9	0	0	0	0
C	4	9	0	0	0	0

Table 2
Fuzzy Data of Variables v_1, v_2, v_3 and v_4 each with Fuzzy sets a, b, c, d and e

			A	B	C
$v_1=$	{	**a**	0	0	0
		b	0	0	2/9
		c	0	1/21	4/9
		d	0	2/21	2/9
		e	1	18/21	1/9
$v_2=$	{	**a**	1/20	0	3/9
		b	0	3/21	4/9
		c	3/20	4/21	2/9
		d	6/20	13/21	0
		e	10/20	1/21	0
$v_3=$	{	**a**	2/20	9/21	1
		b	3/20	6/21	0
		c	6/20	5/21	0
		d	6/20	1/21	0
		e	3/20	0	0
$v_4=$	{	**a**	13/20	16/21	1
		b	7/20	5/21	0
		c	0	0	0
		d	0	0	0
		e	0	0	0

Table 3
Fuzziness of the Student Groups A, B and C in the Variables v_1, v_2, v_3 and v_4

	A	B	C
v_1	0	0.312	0.791
v_2	0.710	0.644	0.659
v_3	0.946	0.751	0
v_4	0.648	0.550	0

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