Illustrating the Continuity of Function Composition

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Abstract

In this paper, a graphical method will be developed that illustrates the fact that: If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.

Introduction

Recently, I had the opportunity to evaluate a colleagues' classroom presentation and the topic of the day was continuity of continuous functions. Partway through the presentation of the proof that the composition of continuous functions is continuous, a student asked for a 'picture' of how the $\varepsilon's$ and $\delta's$ were related. The instructor, who happens to be an excellent analyst, confessed that he did not have knowledge of such an illustration. I later found it was not difficult to make use of a graphical technique for function composition [1] to illustrate this proof.

Continuity of a function is a fundamental concept that either directly or indirectly is addressed early in the mathematical education of our students. The idea the graph of a function being connected is explored as soon as students begin to graph simple functions such as lines in the plane. Later, students increase their understanding of continuity as they explore rational functions and trigonometric functions. Formal definitions of continuity are typically introduced in introductory calculus textbooks [for example 4, 5]. These $\varepsilon - \delta$ definitions are typically graphically illustrated at this level; however, some instructors go past that and educate their students at a level where the definition is used explicitly. For many students this higher level of instruction is reserved for an introductory real analysis or advanced calculus course. At all levels, illustrations can be of benefit to the student.

I have now used a graphical technique several times in first semester calculus courses to indicate how the proof would work. The technique has also been applied in more advanced courses. In all cases, students seem to understand the result much better than when they are instructed without the illustration. After a thorough search, I am convinced that this type of illustration does not appear in literature associated with calculus, advanced calculus, or introductory analysis.

Graphical Composition

As in (Davis 2000), let $(f \circ g)(x) = f(g(x))$ denote the composition of two real valued functions $f(x)$ and $g(x)$. In order to evaluate $f(g(x))$ graphically for the value $x = a$:

On the same set of axes draw the graphs of $y = f(x)$, $y = g(x)$, and $y = x$.

Draw a vertical line from the point $x = a$ on the *x*-axis to the point $(a, g(a))$ on the graph of $y = g(x)$.

Draw a horizontal line from $(a, g(a))$ to the point $(g(a), g(a))$ on the line $y = x$.

Draw a vertical line from $(g(a), g(a))$ to the point $(g(a), f(g(a)))$ on the graph of $y = f(x)$.

Draw a horizontal line from $(g(a), f(g(a)))$ to the point $(0, f(g(a)))$ on the *y*-axis.

Continuity of Composed Functions

A typical proof of the continuity of composed functions is as follows [for example 2, 3].

Since g is continuous at a, $g(a)$ is defined; likewise, f is continuous at $g(a)$, so $f(g(a)) = (f \circ g)(a)$ is defined. At this point, using the instructions above, a picture of $f(g(a))$ is drawn, see Figure 1.

An $\varepsilon - \delta$ proof is now used to show $\lim_{x \to a}(f \circ g)(x) = f \circ g(a)$; i.e., given $\varepsilon > 0$, $\delta > 0$ must be found to satisfy:

If $|x - a| < \delta$, then $|f(g(x)) - f(g(a))| < \varepsilon$.

Due to the continuity of f at $g(a)$, we know there is a δ_1, such that:

If $|z - g(a)| < \delta_1$, then $|f(z) - f(g(a))| < \varepsilon$.

Hence when $g(x)$ is within δ_1 of $g(a)$, then $f(g(a))$ is within ε of $f(g(a))$; i.e.,

If $|g(x) - g(a)| < \delta_1$, then $|f(g(z)) - f(g(a))| < \varepsilon$.

Since g is continuous at a, there is a $\delta > 0$, such that:

If $|x - a| < \delta$, then $|g(x) - g(a)| < \delta_1$.

Chaining inequalities together completes the proof:

$|x - a| < \delta \Rightarrow |g(x) - g(a)| < \delta_1 \Rightarrow |f(g(x)) - f(g(a))| < \varepsilon$.

The Illustration

To draw the corresponding illustration, the following procedure is carried out:

As in Figure 2, draw an $\varepsilon-$neighborhood, on the *y*-axis, about the point $f(g(a))$.

As in Figure 3, draw horizontal lines from the boundaries of this neighborhood to the curve $y = f(x)$.

As in Figure 4, draw two vertical lines from these locations to the line $y = x$. Since the $\varepsilon-$neighborhood includes $f(g(a))$, the vertical lines intersect the line $y = x$ above and below the value of $g(a)$.

As in Figure 5, draw a δ_1-neighborhood about $g(a)$, that when extended to the line $y = x$, remains inside the region associated with the $\varepsilon-$neighborhood.

As in Figure 6, extend the δ_1-neighborhood horizontally to the curve $y = g(x)$.

As in Figure 7, draw horizontal lines from these boundaries to the *x*-axis. These lines will define an interval that has $x = a$ in its interior.

As in Figure 8, draw a $\delta-$neighborhood about $x = a$, that remains inside the previous interval.

As in Figure 9, the boundaries of the $\delta-$neighborhood are graphically evaluated by g and then f.

The resulting neighborhood on the *y*-axes remains within ε of $f(g(a))$ as desired and the illustration is complete.

Conclusion

I have found that this method of illustrating that the composition of continuous functions is continuous requires no significant additional classroom time. It has proven to be a very helpful teaching aid in the classroom. Students seem to have a better understanding of both function composition as well as the associated continuity properties. I also find that students who have been exposed to this graphical method are more likely to be able to understand how to make choices for ε and/or δ in specific computational exercises. I expect that others will find this type of illustration helpful as well.

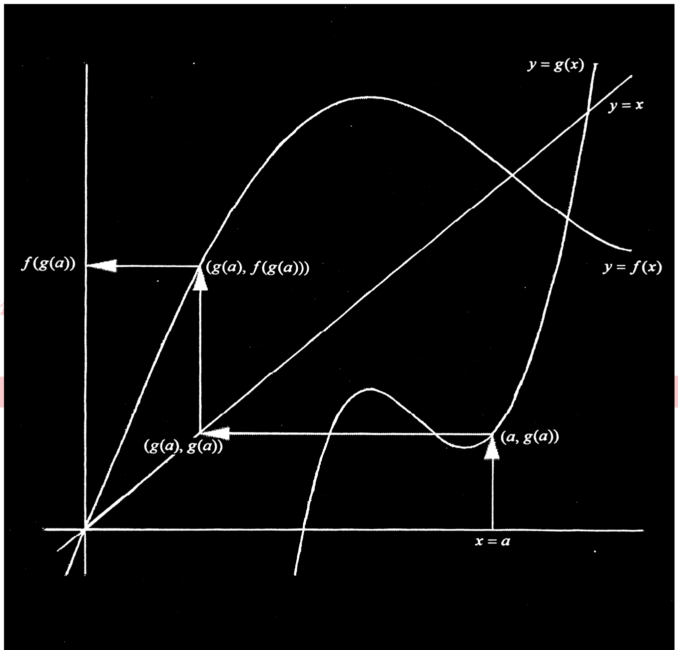

Figure 1

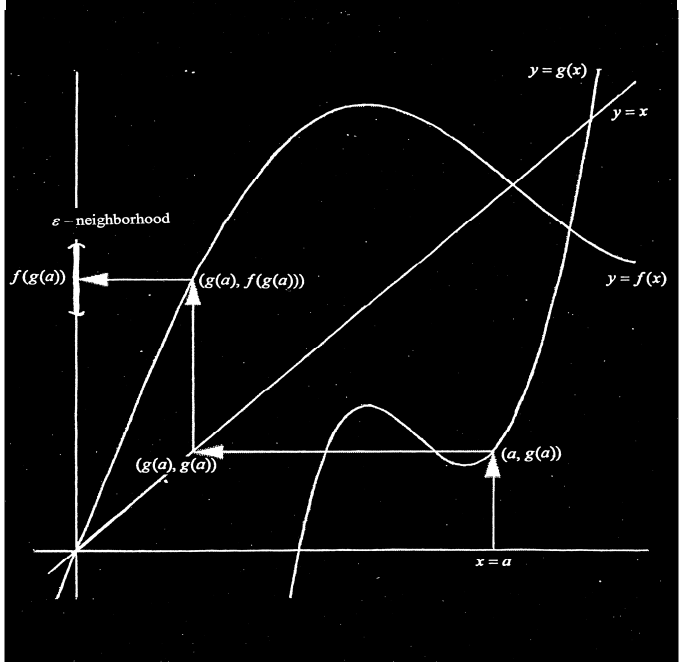

Figure 2

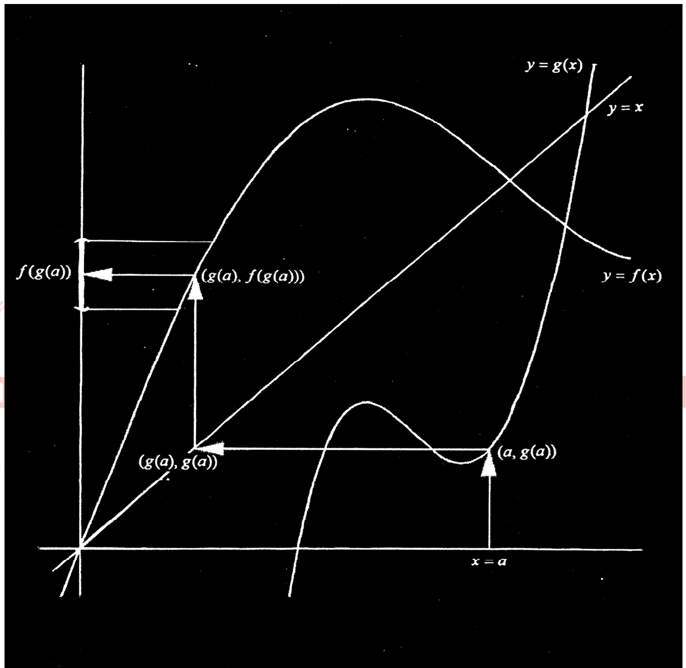

Figure 3

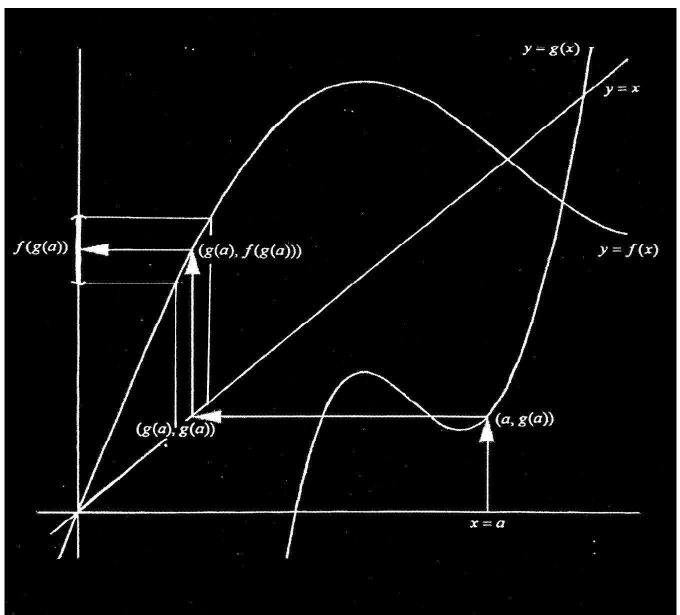

Figure 4

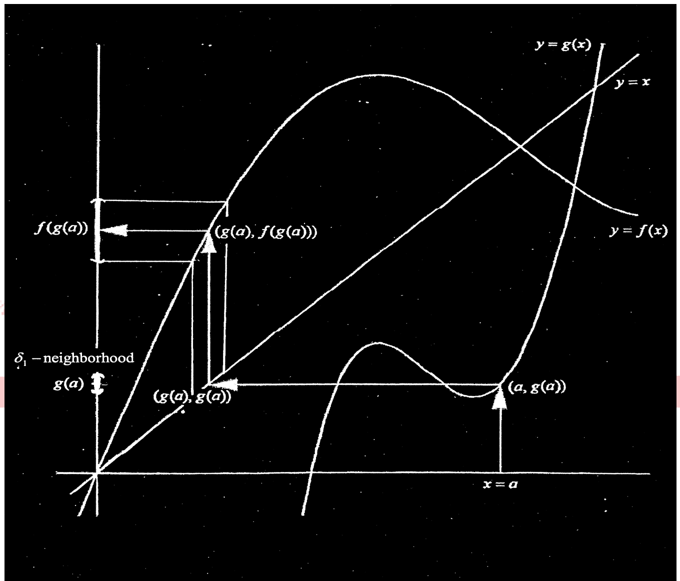

Figure 5

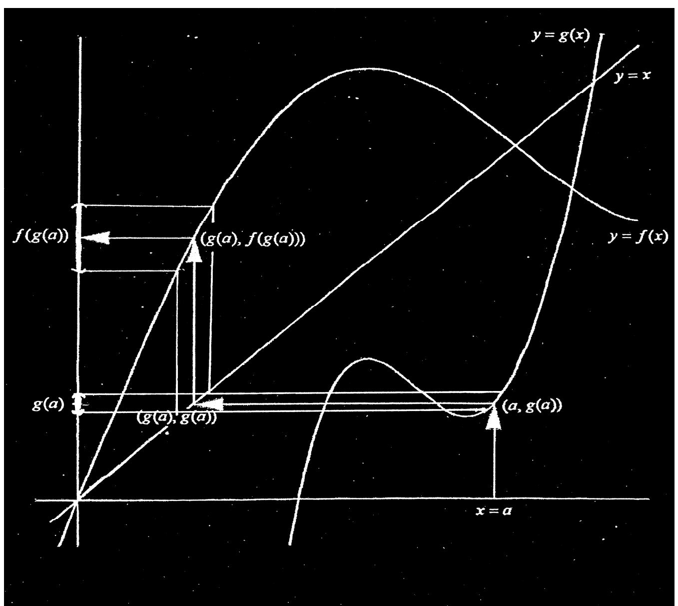

Figure 6

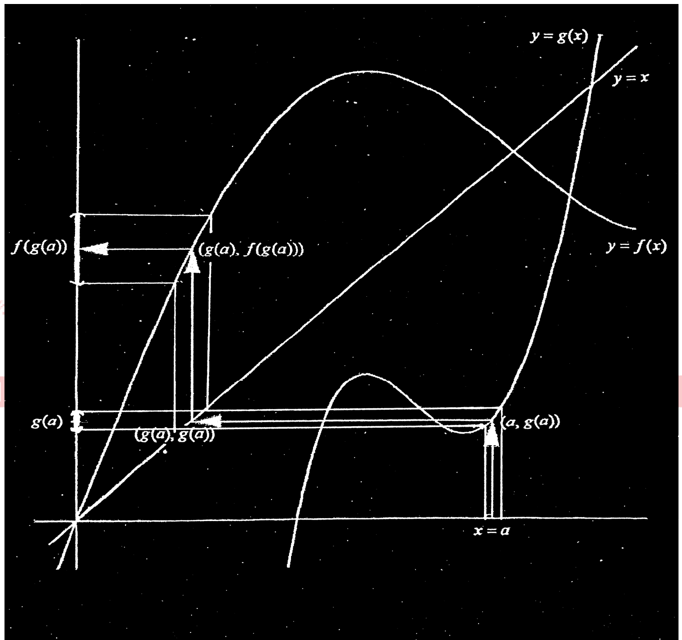

Figure 7

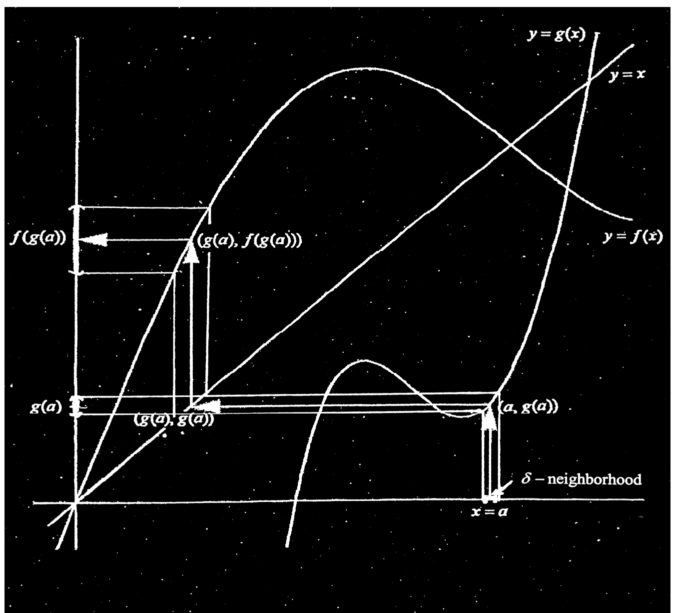

Figure 8

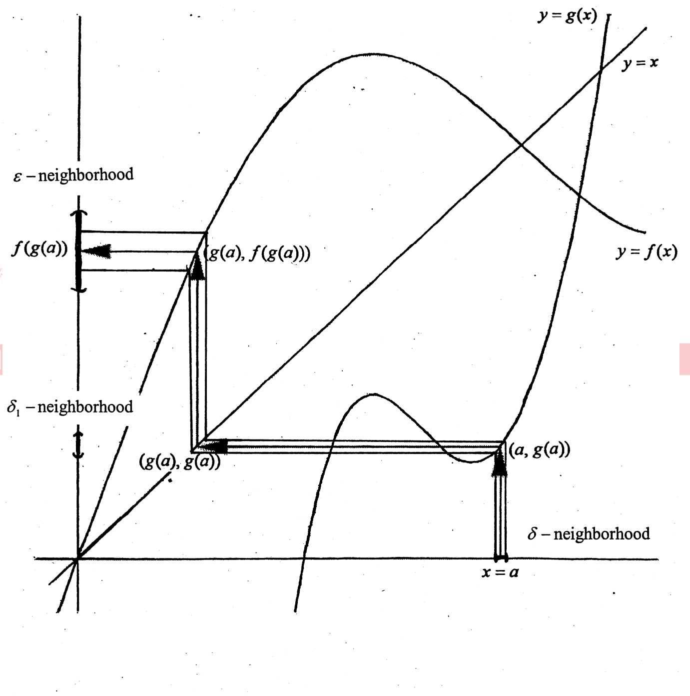

Figure 9

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References

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