Derivatives without Limits: Application to a Quadratic Function Divided by a Linear Function

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Abstract

It is shown that the method of finding the slopes of the tangents to some special functions, without resorting to the traditional method of calculus using limits, can be extended to a wider class of functions than those considered by M. Wahl. Indeed, his method is shown to be valid for a quadratic function divided by a linear function.

Introduction

As pointed out by M. Wahl [1] and a relevant website [2], the derivatives of some common functions can be found without resorting to the calculus concept of limits. Specifically, the functions considered in [1] are quadratics, reciprocals, square roots and ellipses. The method uses the fact that the tangent line intercepts the curve at exactly one point for these functions. However, these are not the only functions for which this is true. Indeed, the purpose of this paper is to show that the method can be applied to a quadratic function divided by a linear function, i.e., $q(x) = \dfrac{ax^2 + bx + c}{dx + e}$: hence, the derivative of this function can be found without the calculus concept of a limit.

Interestingly, in the special case of $d = 0$ and $e = 1$, $q(x)$ is the expression for a quadratic. Furthermore, in the case of $a = b = 0$, $q(x)$ is the expression for a reciprocal. Therefore, finding the derivative of $q(x)$ generalizes some of the results in [1].

Derivatives of $q(x)$ without Limits

Finding the derivative of a function requires that the slope of the tangent to that function at a given point be found. The tangent line has equation

$$y = mx + B, \tag{1}$$

where m is the slope of the tangent, i.e. the derivative of the function.

Furthermore, the tangent line, for this function, intersects the curve at exactly one point, as illustrated in Fig.1.Hence,

$$mx + B = \frac{ax^2 + bx + c}{dx + e} \tag{2}$$

has exactly one solution.

Rearranging (2) gives

$$\begin{aligned}
(mx + B)(dx + e) &= ax^2 + bx + c \\
mdx^2 + (em + Bd)x + eB &= ax^2 + bx + c \\
(md - a)x^2 + (em + Bd - b)x + eB - c &= 0 \\
x^2 + \frac{em + Bd - b}{md - a}x + \frac{eB - c}{md - a} &= 0.
\end{aligned} \tag{3}$$

Using the quadratic formula to solve (3) gives

$$x = -\frac{1}{2}\left(\frac{em + Bd - b}{md - a}\right) \pm \frac{1}{2}\sqrt{\left(\frac{em + Bd - b}{md - a}\right)^2 - 4\left(\frac{eB - c}{md - a}\right)}. \tag{4}$$

Hence, in order for (4) to have exactly one solution, we must have

$$x = -\frac{1}{2}\left(\frac{em + Bd - b}{md - a}\right) \tag{5}$$

and

$$\left(\frac{em + Bd - b}{md - a}\right)^2 - 4\left(\frac{eB - c}{md - a}\right) = 0. \tag{6}$$

Rearranging (6) gives

$$(em + Bd - b)^2 - 4(eB - c)(md - a) = 0. \tag{7}$$

From (5),

$$4x^2 (md - a)^2 = (em + Bd - b)^2 \tag{8}$$

Substituting (8) into (7) and dividing by $4(md - a)$ produces

$$x^2 (md - a) - (eB - c) = 0. \tag{9}$$

Solving (9) for B gives

$$B = \frac{x^2\left(md - a\right) + c}{e}. \tag{10}$$

Substituting (10) into (5) produces

$$x = -\frac{1}{2}\left(\frac{em + \left[\dfrac{x^2\left\{md - a\right\} + c}{e} \right] d - b}{md - a} \right)$$

$$= -\frac{1}{2}\left(\frac{e^2 m + x^2 d\left\{md - a\right\} + cd - eb}{e\left\{md - a\right\}} \right). \tag{11}$$

Rearranging (11) results in

$$\begin{aligned} 2xe\left\{md - a\right\} &= -\left(e^2 m + x^2 d\left\{md - a\right\} + cd - eb\right) \\ 2xemd - 2xea &= -e^2 m - x^2 d^2 m + ax^2 d - cd + eb \\ \left(2xed + e^2 + x^2 d^2\right) m &= ax^2 d - cd + eb + 2xea. \end{aligned} \tag{12}$$

Solving (12) for m gives the desired derivative of $q(x)$. Therefore,

$$m = \frac{adx^2 + 2aex + be - cd}{2xed + e^2 + x^2 d^2} = \frac{adx^2 + 2aex + be - cd}{\left(dx + e\right)^2}. \tag{13}$$

Fortunately, this is the same result that is gotten from calculus using the quotient rule, i.e.

$$m = \frac{\left(2ax + b\right)\left(dx + e\right) - \left(ax^2 + bx + c\right)d}{\left(dx + e\right)^2}. \tag{14}$$

Expanding the numerator in (14) shows that it is the same as that of (13).

As mentioned above, when $d = 0$ and $e = 1$, $q(x)$ is the expression for a quadratic and (13) becomes $m = 2ax + b$, i.e., the derivative of a quadratic, as required. On the other hand, if $a = b = 0$, $q(x) = \dfrac{c}{dx+e}$ is the expression for a reciprocal, and (13) becomes $m = -\dfrac{cd}{(dx+e)^2}$, the derivative of the reciprocal, as required. Therefore, it is clear that finding the derivative of $q(x) = \dfrac{ax^2+bx+c}{dx+e}$ without limits, generalizes the results of [1].

Conclusion

It has been shown that the derivative of $q(x) = \dfrac{ax^2+bx+c}{dx+e}$ can be found without limits. This means that the class of functions for which this has been shown to be possible has been increased over that available in [1]. In fact, the quadratic and reciprocal cases of [1] are just special cases for the function studied in this paper.

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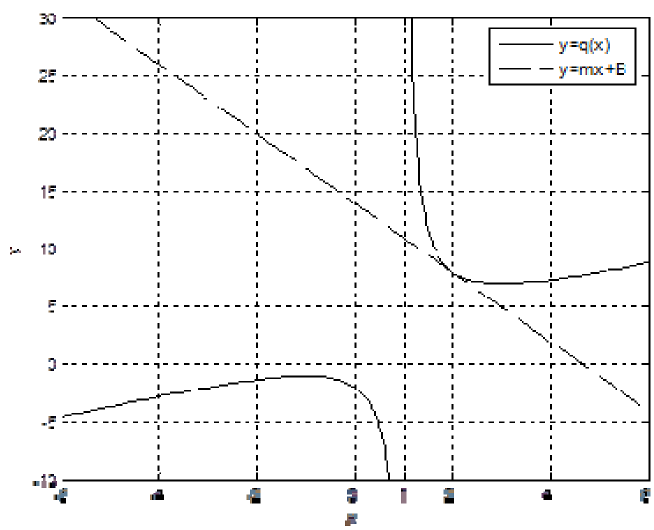

Fig.1. Typical plot of $q(x) = \dfrac{ax^2+bx+c}{dx+e}$ and its tangent at $x = 2$. For this specific plot, $q(x) = \dfrac{x^2+x+2}{x-1}$ along with the tangent line $y = -3x+14$.

References

[1] M. Wahl, “Derivatives without Limits,” *Math Horizons*, pg. 12 and pg. 30, Nov. 2008.

[2] CTK Wiki Math, “Derivatives without Limits,” http://www.cut-the-knot.org/wiki-math/index.php?n=Calculus.DerivativesWithoutLimits, accessed 20th Nov., 2010.