Explicit Solutions for Transcendental Equations: A Technical Note

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Abstract

A method based on Cauchy's integral theorem to formulate the roots of an analytic transcendental function is applied to the solution of some transcendental equations: exp (-b z) = z, exp (5-z) +.01 z =.55 and tan z = h z. It represents a simple and fast way to solve analytical transcendental equations.

Introduction

Explicit expressions for the roots of any analytic transcendental function can be obtained by means of the Cauchy's integral theorem [1].
Let us consider a function, f(z), obtained from a transcendental equation with single singularity at z_o somewhere inside region C and analytic elsewhere in the region. The singularity can be removed by multiplying f(z) by $(z - z_0)$. Cauchy's theorem implies that the path integral of the new function around closed curve C must be zero:

(1) $\oint_c (z - z_o) f(z)\, dz = 0$

Then, from it results that

$$\oint_c z\, f(z)\, dz - z_0 \oint_c f(z)\, dz = 0$$

$$z_0 \oint_c f(z)\, dz = \oint_c z\, f(z)\, dz$$

That let us locate the singularity

(2) $z_o = \dfrac{\oint_c z\, f(z) dz}{\oint_c f(z) dz}$

In fact, one way to evaluate (2) uses a circle in the complex plane that circumscribes the root. The closed curve C may then be described as a circle in the complex plane with center h and radius R as long as the circle contains the root

(3) $z = h + Re^{i\theta}$, $dz = iRe^{i\theta} d\theta$.

It is possible to determine the roots enclosed by the path C. Substituting $z = h + Re^{i\theta}$, equation (2) results

$$z_o = \frac{\oint_c (h + Re^{i\theta})\, f(h + Re^{i\theta})\, d(h + Re^{i\theta})}{\oint_c f(h + Re^{i\theta})\, d(h + Re^{i\theta})}$$

$$z_o = h + R\,\frac{\oint_c e^{i\theta} f(h + Re^{i\theta})\,d(h + Re^{i\theta})}{\oint_c f(h + Re^{i\theta})\,d(h + Re^{i\theta})}$$

(4) $$z_o = h + \left[\frac{\int_0^{2\pi} \omega(\theta) e^{i2\theta}\,d\theta}{\oint_0^{2\pi} \omega(\theta) e^{i\theta}\,d\theta}\right]$$

where

(5) $\omega(\theta) = f(h + R\,e^{i\theta})$,

The nth coefficient in a complex exponential Fourier series is

(6) $$C_n = \frac{1}{2\pi}\int_0^{2\pi} f(t)\; e^{in\theta}\,dt$$

where f(t) is the function to be represented by the series.
We can see that the term in brackets in (4) is equal to the ratio of the second Fourier series coefficient to the first for the function $\omega(\theta)$.
Fourier series coefficients may be calculated easily using standard complex fast Fourier transform (FFT) functions found in most mathematical software packages. If f(z) is analytic at h, multiplying f(z) by a factor of $(z - h) = Re^{i\theta}$ will not change the location of the singularities of f(z). This implies that for a given singularity the term in brackets is also equal to any ratio of the (j + 1) to the jth Fourier series coefficients of $\omega(\theta)$, with $j \geq 1$. [1]

Some results with these explicit solutions of transcendental equations.

a) Explicit roots of the transcendental equation $e^{-bZ} = z$, b is an integer greater than 1.

The function z intersects e^{-bZ}, for $z > 0$.
Reordering the transcendental equation we can obtain a function with a singularity

(7) $$f(z) = \frac{1}{e^{-bz} - z}$$

Replacing z according (3), we get

$$\omega(\theta) = f(h + R\,e^{i\theta}) = \frac{1}{e^{-b(h+R\,e^{i\theta})} - (h + Re^{i\theta})}$$

Assigning R=1 and h=0, becomes $z = h + R\,e^{i\theta} = e^{i\theta}$

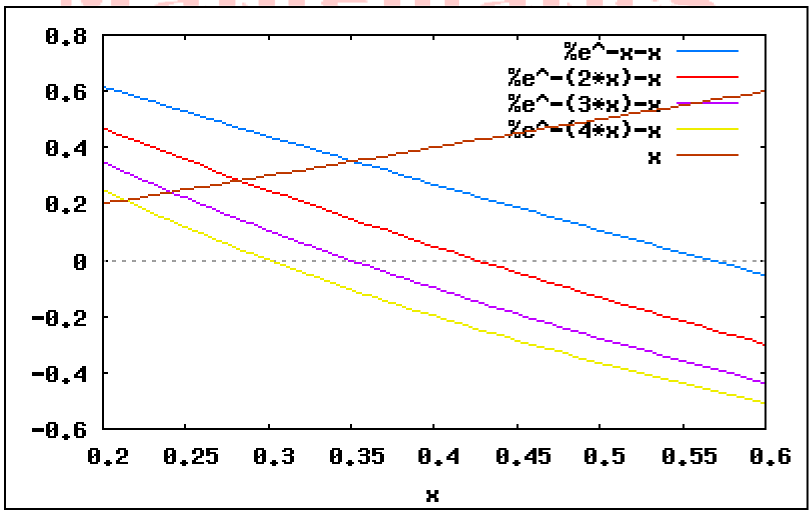
The singularity corresponding to the equation (2), results

$$z_0 = \frac{\int_0^{2\pi} \omega(\theta) e^{i2\theta}\,d\theta}{\oint_0^{2\pi} \omega(\theta) e^{i\theta}\,d\theta}, \text{ being } \omega(\theta) = \frac{1}{e^{-b(e^{i\theta})} - e^{i\theta}},$$

It can be described by the coefficients C_1 and C_2 as $Z_0 = \frac{C_2}{C_1}$, given by (6)

b	1	2	3	4
C_1	**-0.6381**	**-0.5398**	**-0.4878**	**-0.4541**
C_2	**-0.3619**	**-0.2301**	**-0.1707**	**-0.1365**
$z=\frac{C_2}{C_1}$	**0.56715**	**0.42627**	**0.34994**	**0.30059**
exp(-b*z)	**0.56714**	**0.42633**	**0.35**	**0.30048**

This table show a total agreement between $z = \frac{C_2}{C_1}$ and exp **(-b*z)** for b from 1 to 4



b) Explicit roots of the transcendental equation $e^{5-z} + \frac{z}{100} - 0.55 = 0$, **[2]**

Reordering the transcendental equation we can obtain again a function with a singularity

(8) $f(z) = \frac{1}{e^{5-z} + \frac{z}{100} - 0.55}$

A good choice for the closed path C is a circle R=5 centered at $h_1 = 5$ and $h_2 = 55$, being $z = h + Re^{i\theta}$,

$$\omega(\theta) = f(h + Re^{i\theta}) = \frac{1}{e^{5-(h+Re^{i\theta})} + \frac{5+Re^{i\theta}}{100} - 0.55}$$

$$\omega(\theta) = \left\{e^{5-h}\, e^{-5e^{i\theta}} + \frac{h+5e^{i\theta}}{100} - 0.55\right\}^{-1}$$

$$\omega(\theta) = \left\{e^{5-h}\, e^{-5(\cos\theta + i\sin\theta)} + \frac{h+5(\cos\theta + i\sin\theta)}{100} - 0.55\right\}^{-1}$$

$$\omega(\theta) = \left\{e^{5-h}\; e^{-5\cos\theta}\; e^{-5i\sin\theta} + \frac{h+5\cos\theta}{100} + \frac{i\sin\theta}{20} - 0.55\right\}^{-1}$$

$$\omega(\theta) = \left\{e^{5-h}\; e^{-5\cos\theta}\,[\cos(5\sin\theta) - i\sin(5\sin\theta)] + \frac{h+5\cos\theta}{100} + \frac{i\sin\theta}{20} - 0.55\right\}^{-1}$$

$$\omega(\theta) = \left[\; e^{5-h}\; e^{-5\cos\theta}\cos(5\sin\theta) + \frac{h+5\cos\theta}{100} - 0.55 - i\left\{e^{5-h}\; e^{-5\cos\theta}\sin(5\sin\theta) + \frac{\sin\theta}{20}\right\}\;\right]^{-1}$$

We will find the Fourier coefficients A_1 and A_2 for h = 5 and h=55.

$$\omega(\theta) = \left[\; e^{5-h-5\cos\theta}\;\cos(5\sin\theta) + \frac{h+5\cos\theta}{100} - 0.55 - i\left\{e^{5-h-5\cos\theta}\;\sin(5\sin\theta) + \frac{\sin\theta}{20}\right\}\;\right]^{-1}$$

The location of the singularity z_o given by (2) is

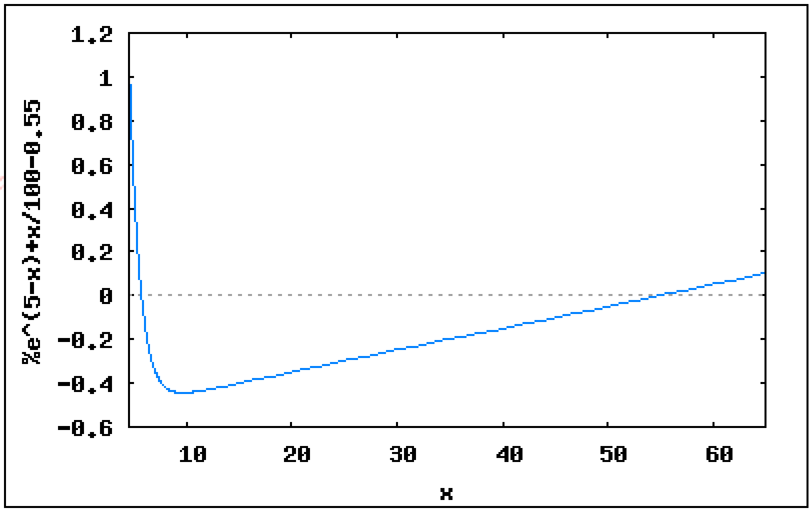
$$z_o = h + R\,\frac{\int_0^{2\pi}\omega(\theta)e^{i2\theta}d\theta}{\int_0^{2\pi}\omega(\theta)e^{i\theta}d\theta}$$

If we consider (6), results

$$z_o = h + R\,\frac{C_2}{C_1},$$

h	5	55
C_1	**-0.4141**	**20**
C_2	**-0.0586**	**0**
Z= h + 5 $\frac{C_2}{C_1}$	**5.70756**	**55**
exp(5-Z)+(Z/100)-0.55	**-8E-05**	**0**

For z = 5.70756 and z = 55 we find values of exp **(5-Z)+(Z/100)-0.55** very close to zero as we expected.



c) Explicit roots of the transcendental equation tan (Z) = hZ, [3]

We take in account that

$$\frac{\sin z}{\cos z} = Bz, \quad \sin z = B z \cos z, \quad \text{then} \quad \sin z - B z \cos z = 0$$

To configure the equation to provide a singularity we use

(9) $f(z) = \frac{1}{\sin(Z) - B Z \cos(Z)}$ (9)

From the graphs of tan z and z results a good choice for the closed path C a circle

$R = \frac{\pi}{4}$ centered at $h_n = \left(n - \frac{3}{4}\right)\pi$.

$z = h_n + R\, e^{i\theta}$

$$Z_{o_n} = h_n + R\, \frac{\oint_c e^{i\theta} f(h_n + Re^{i\theta}) e^{i2\theta} d\theta}{\oint_c f(h_n + Re^{i\theta}) e^{i\theta} d\theta}$$

$$\omega_n(\theta) = \frac{1}{\sin(h_n + R\, e^{i\theta}) - B\,(h_n + R\, e^{i\theta})\cos(h_n + R\, e^{i\theta})}$$

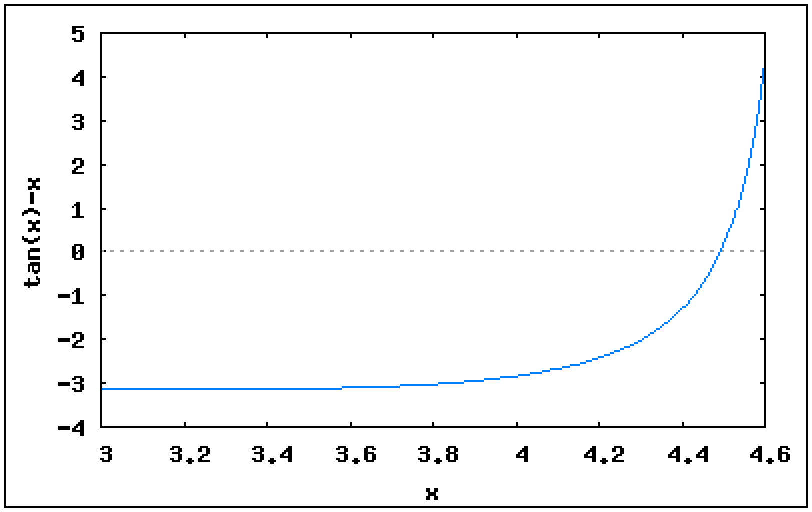
$B = 1, and\ R = \frac{\pi}{4}$

$$\omega_n(\theta) = \{\sin\left(h_n + \frac{\pi}{4} e^{i\theta}\right) - (h_n + \frac{\pi}{4} e^{i\theta}) \cos(h_n + \frac{\pi}{4} e^{i\theta})\}^{-1}$$

In this way the singularity $Z_{o_n} = h_n + \frac{\pi}{4} \frac{\int_0^{2\pi} \omega_n(\theta) e^{i2\theta} d\theta}{\oint_0^{2\pi} \omega_n(\theta) e^{i\theta} d\theta} = h_n + \frac{\pi}{4} \frac{c_2}{c_1}$,

h	5π/4
C_1	**-0.2903**
C_2	**-0.2094**
Z=h+(π/4) $\frac{C_2}{C_1}$	**4.49338**
tan(z)	**4.49287**

This table show that **tan(z) and Z=h+(π/4)** $\frac{A_2}{A_1}$ are in good agreement.



Concluding Remarks

This method can be applied to solutions of any transcendental equation when we reorder an analytical function to have a singularity and applying the Cauchy's theorem to obtain the value of the singularity that represents the solution of the transcendental equation. In this description some coefficients in a complex exponential Fourier series of the function obtained contribute to simplify the result. It is important to highlight that it is an easy and fast way to solve them.

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