Fuzzy Logic and Uncertainty in Problem-Solving

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Abstract

We develop a general fuzzy model that can be adapted properly to represent several didactic situations in Mathematics Education characterized by a degree of fuzziness and/or uncertainty and we apply it in particular to describe in a more effective way the problem-solving process. Two classroom experiments are also presented illustrating the use of our model in practice.

Introduction

They appear often didactic situations in Mathematics Education characterized by a degree of fuzziness and/or uncertainty (e.g. problem-solving, mathematical modelling, learning, mathematics, etc). In fact, students' cognition utilizes in general concepts that are inherently graded and therefore fuzzy. On the other hand, from teacher's point of view there usually exists vagueness about the degree of success of students in each of the stages of the corresponding didactic situation. All these gave us the impulsion to introduce principles of *fuzzy logic* and of *uncertainty theory* in order to describe in a more effective way the process of such kind of situations in classroom.

The concept of uncertainty, which emerges naturally within the broad framework of fuzzy sets theory, is involved in any didactic situation, especially when dealing with real-world problems. Uncertainty is a result of some information deficiency. In fact, information pertaining to the model within which a real situation is conceptualized may be incomplete, fragmentary, not full reliable, vague, contradictory, or deficient in some other way. Thus the amount of information obtained by an action can be measured in general by the reduction of uncertainty resulting from the action. In other words the amount of uncertainty regarding some situation represents the total amount of potential information in this situation.

In the next we shall develop a general fuzzy model that can be adapted in each case in order to represent the process of the corresponding didactic situation and we shall apply it in particular to represent the problem-solving process in classroom.

For general facts on fuzzy sets and uncertainty theory we refer freely to Klir and Folger (1988).

The General Model

Let us consider a group of n students, $n \geq 2$, in classroom. Denote by S_i , i=1,2,3 , the main stages of the process of the didactic situation that we want to represent and by *a, b, c, d,* and *e* the linguistic labels of negligible, low, intermediate, high and complete success respectively of a student in each of the S_i's.

Set $U = \{a, b, c, d, e\}$. We are going to attach to each stage S_i a fuzzy subset, A_i of U. For this, if n_{ia}, n_{ib}, n_{ic}, n_{id} and n_{ie} denote the number of students that faced negligible, low, intermediate, high and complete success at stage S_i respectively, i=1,2,3, we define the *membership function* m_{Ai} for each x in U, as follows:

$$m_{A_i}(x) = \begin{cases} 1, & \text{if } \frac{4n}{5} < n_{ix} \leq n \\ 0{,}75, & \text{if } \frac{3n}{5} < n_{ix} \leq \frac{4n}{5} \\ 0{,}5, & \text{if } \frac{2n}{5} < n_{ix} \leq \frac{3n}{5} \\ 0{,}25, & \text{if } \frac{n}{5} < n_{ix} \leq \frac{2n}{5} \\ 0, & \text{if } 0 \leq n_{ix} \leq \frac{n}{5} \end{cases}$$

Then the fuzzy subset A_i of U corresponding to S_i has the form:

$$A_i = \{(x, m_{Ai}(x)) : x \in U\},\ i=1, 2, 3.$$

In order to represent all possible student *profiles (overall states)* during the corresponding process we consider a *fuzzy relation*, say R, in U^3 of the form

$$R = \{(s, m_R(s)) : s=(x, y, z) \in U^3\}.$$

We make the hypothesis that the stages of the process of the corresponding didactic situation are depended to each other. This means that the degree of success of a student in a certain stage depends upon the degree of his/her success in the previous stages, as it usually happens in practice. Under this hypothesis and in order to determine properly the membership function m_R we give the following definition:

A profile s=(x, y, z), with x, y, z in U, is said to be *well ordered* if x corresponds to a degree of success equal or greater than y, and y corresponds to a degree of success equal or greater than z.

For example, *(c, c, a)* is a well ordered profile, while *(b, a, c)* is not.

We define now the *membership degree* of a profile *s* to be

$m_R(s)=m_{A_1}(x)m_{A_2}(y)m_{A_3}(z)$, if *s* is well ordered, and *0* otherwise. In fact, if for example profile *(b, a, c)* possessed a nonzero membership degree, how it could be possible for a student, who has failed during the middle stage, to perform satisfactorily in the next stage?

In the next, for reasons of brevity, we shall write m_s instead of $m_R(s)$. Then the *possibility* r_s of profile s is defined by $r_s = \frac{m_s}{\max\{m_s\}}$, where $max\{m_s\}$ denotes the maximal value of m_s , for all s in U^3. In other words the possibility of s expresses the "relative membership degree" of s with respect to *max* $\{m_s\}$.

As we have seen above the amount of information obtained by an action can be measured by the reduction of uncertainty resulting from the action. Accordingly students' uncertainty during the process of the corresponding didactic situation is connected to students' capacity in obtaining relevant information. Therefore a measure of uncertainty could be adopted as a measure of students' capacities. Within the domain of possibility theory uncertainty consists of *strife (or discord)*, which expresses conflicts among the various sets of alternatives, and *non-specificity (or imprecision)*, which indicates that some alternatives are left unspecified, i.e. it expresses conflicts among the sizes (cardinalities) of the various sets of alternatives (Klir 1995, p.28). Strife is measured by the function $ST(r)$ on the ordered possibility distribution r: $r_1=1\geq r_2 \geq \ldots\ldots \geq r_n \geq r_{n+1}$ of the student group defined by $ST(r) = \frac{1}{\log 2}[\sum_{i=2}^{n}(r_i - r_{i+1})\log\frac{i}{\sum_{j=1}^{i} r_j}]$, while non-specificity is measured by the function

$$N(r) = \frac{1}{\log 2}[\sum_{i=2}^{n}(r_i - r_{i+1})\log i\ ].$$

The sum $T(r)=ST(r)+N(r)$ is a measure of the *total possibilistic uncertainty* for ordered possibility distributions. Therefore the total possibilistic uncertainty of the student group during the process can be adopted as a measure of students' capacities. This is reinforced by Shackle (1961) who argues that human reasoning can be formalized more adequately by possibility theory rather, than by probability theory. The lower is the value of $T(r)$ (which means greater reduction of the initially existing uncertainty), the better the performance of the student group during the process of the corresponding didactic situation.

Assume finally that one wants to study the *combined results of behaviour* of k different student groups, $k\geq 2$, during the same process. For this we introduce the *fuzzy variables* $A_1(t)$, $A_2(t)$ *and* $A_3(t)$ with $t=1, 2,\ldots, k$. The values of these variables represent fuzzy subsets of U corresponding to the stages of the process for each of the k student groups; e.g. $A_1(2)$ represents the fuzzy subset of U corresponding to the stage of planning for the second group (t=2). It becomes evident that, in order to measure the degree of evidence of combined results of the k groups, it is necessary to define the possibility $r(s)$ of each student profile s with respect to the membership degrees of s for all student groups. For this reason we introduce the *pseudo-frequencies* $f(s) = \sum_{t=1}^{k} m_s(t)$ and we define $r(s) = \frac{f(s)}{\max\{f(s)\}}$, where *max* $\{f(s)\}$ denotes the maximal pseudo-frequency. Obviously the same method could be applied when one wants to study the combined results of behaviour of a student group during k different didactic situations.

A Fuzzy Model for Problem-Solving

In earlier papers we have adapted properly the above general model in order to represent the processes of Learning Mathematics (Voskoglou 2009a), of Mathematical Modelling (Voskoglou 2010, 2011), of Case-Based Reasoning (Voskoglou 2009c), as well as for several applications in Management (Voskoglou 2003). Here we are going to do the same for the process of Problem-Solving.

History of Problem-Solving

Problem–Solving (P-S) is a principal component of mathematics education with a long history and has supported numerous research programs at all levels. Given the importance of P-S, the orientations and structure of many curriculum proposals and teaching models throughout the world have been either directly or indirectly influenced by it.

In earlier papers (Voskoglou 2007a, 2008), we have examined the role of P-S in learning mathematics and we have attempted a review of the progress of research on P-S in mathematics education from the time that Polya presented his first ideas on the subject until nowadays. Here is a rough chronology of that progress:

1950's – 1960's: Polya's ideas on the use of heuristic strategies in P-S (Polya 1945, 1954, 1963, 1962/65)

1970's: Emergency of mathematics education as a self – sufficient science (research methods were almost exclusively statistical). Research on P-S was mainly based on Polya's ideas.

1980's: A framework describing the P-S process, and reasons for success or failure in P-S, e.g. Schoenfeld 1980, 1985b, Lester, Garofalo & Kroll 1989, etc.

1990's: Models of teaching using P-S, e.g. constructivist view of learning (see Voskoglou 2007c and its references), Mathematical modelling and applications (see Voskoglou 2006 and its references), etc.

2000's: While early work on P-S focused mainly on analyzing the P-S process and on describing the proper heuristic strategies to be used in each of its stages, more recent investigations have focused mainly on solvers' behaviour and required attributes during the P-S process; e. g. MPS Framework of Carlson and Bloom (2005), theory of goal-directed behaviour , (Schoenfeld 2007), etc.

The Multidimensional Problem Solving Framework of Carlson and Bloom

Carlson and Bloom (2005) drawing from the large amount of literature related to P-S developed a broad taxonomy to characterize major P-S attributes that have been identifying as relevant to P-S success. This taxonomy gave genesis to their *Multidimensional Problem-Solving Framework (MPSF),* which includes four phases: *Orientation, Planning, Executing* and *Checking*. It has been observed that once the solvers oriented themselves to the problem space, the plan-execute-check cycle was usually repeated through out the remainder of the solution process; only in a few cases a solver obtained linearly the solution of a problem (i.e. he/she made this cycle only once). Thus embedded in the framework are two cycles (one cycling back and one cycling forward), each of which includes the three out of the four phases, that is planning, executing and checking. It has been also observed that, when contemplating various solution approaches during the planning phase of the P-S process, the solvers were at times engaged in a *conjecture-imagine-evaluate (accept/reject)* sub-cycle. Therefore, apart of the two main cycles, embedded in the framework is the above sub-cycle, which is connected to the phase of planning (Carlson and Bloom 2005, Figure 1).

It is of worth to notice that there are many similarities among the five stages of Schoenfeld's expert performance model (Schoenfeld 1980) and the four phases of MPSF. In fact, the stage of analysis of the problem of Shoenfeld's model corresponds to the phase of orientation, the stage of design corresponds to the phase of planning, the stage of exploration corresponds to the conjecture-imagine-evaluate sub-cycle connected to the phase of planning, the implementation of the solution corresponds to the phase of executing and finally the stage of verification corresponds to the phase of checking. The qualitative difference between these two models is actually that, while the former focuses on describing the P-S process and the proper heuristic strategies to be used in each of its stages, the latter focuses on solver's behaviour and required attributes during the P-S process (Voskoglou 2008; section 4).

The model

The construction of our fuzzy model for the P-S process is based on MPSF. For this, we consider a group of n students, $n \geq 2$, in classroom during the P-S process and we denote by S_i, i=1, 2, 3 the phases of planning, executing and checking. To each of the S_i's we attach a fuzzy subset, say A_i, of U by defining the membership function m_{Ai} exactly as we have described in our general model. Notice that in the same way one could also attach to the phase of orientation a fuzzy subset of U. However this, although it makes the presentation of our fuzzy model technically much more complicated, it is not so important, since orientation, although it deserves some attention, is actually an introductory step of the P-S process that could be considered as a sub-phase of planning. The above manipulation is a simplification made to the real system in order to transfer from it to the *"assumed real system"*. This is a standard technique applied during the modelling process of real world problems, which enables the formulation of them in a form ready for mathematical treatment (Voskoglou 2007b, section 1).The development of the rest of the model relies upon the lines of our general model presented above.

Classroom applications

The following two experiments performed recently at the Graduate Technological Educational Institute (T.E.I.) of Patras in Greece. In the first of them our subjects were 35 students of the School of Technological Applications, i.e. future engineers, and our basic tool was a list of 10 problems (see Appendix) given to students for solution (time allowed 3 hours). Before starting the experiment we gave the proper instructions to students emphasizing among the others that we are interested for <u>all their efforts</u> (successful or not) during the P-S process, and therefore they must keep records on their papers for all of them, at <u>all stages</u> of the P-S process. This manipulation enabled as in obtaining realistic data from our experiment for each stage of the P-S process and not only those based on students' final results that could be obtained in the usual way of graduating their papers.

Our characterizations of students' performance at <u>each stage</u> of the P-S process involved:

- Negligible success, if they obtained (at the particular stage) positive results for less than 2 problems.
- Low success, if they obtained positive results for 2, 3, or 4 problems.
- Intermediate success, if they obtained positive results for 5, 6, or 7 problems.
- High success, if they obtained positive results for 8, or 9 problems.
- Complete success, if they obtained positive results for all problems.

Examining students' papers we found that 15, 12 and 8 students had intermediate, high and complete success respectively at stage of planning. Therefore we obtained that $n_{1a}=n_{1b}=0$, $n_{1c}=15$, $n_{1d}=12$ and $n_{1e}=8$. Thus, by the definition of $m_{A_i}(x)$, planning corresponds to a fuzzy subset of U of the form:

$A_1 = \{(a,0),(b,0),(c, 0.5),(d, 0.25),(e, 0.25)\}$.

In the same way we represented the stages of executing and checking as fuzzy sets in U by $A_2 = \{(a,0),(b,0),(c, 0.5),(d, 0.25),(e,0)\}$ and

$A_3 = \{(a, 0.25),(b, 0.25),(c, 0.25),(d,0),(e,0)\}$ respectively.

Using the definition given in section 2 we calculated the membership degrees of the 5^3 (ordered samples with replacement of 3 objects taken from 5) in total possible students' profiles (see column of $m_s(1)$ in Table 1). For example, for $s=(c, c, a)$ one finds that

$m_s = m_{A_1}(c).\ m_{A_2}(c).\ m_{A_3}(a) =(0.5).(0.5).(0.25)=0.06225$.

It turns out that (c, c, a) was one of the profiles of maximal membership degree and therefore the possibility of each s in U^3 is given by $r_s=\dfrac{m_s}{0{,}06225}$.

Calculating the possibilities of all profiles (see column of $r_s(1)$ in Table 1) one finds that the ordered possibility distribution for the student group is:

$r_1=r_2=1, r_3=r_4=r_5=r_6=r_7=r_8=0{,}5, r_9=r_{10}=r_{11}=r_{12}=r_{13}=r_{14}=0{,}258,$
$r_{15}=r_{16}=\ldots\ldots=r_{125}=0$. Thus using a calculator we found that

$$\text{ST}(r) = \frac{1}{\log 2}\left[\sum_{i=2}^{14}(r_i - r_{i+1})\log\frac{i}{\sum_{j=1}^{i} r_j}\right]$$

$$\approx \frac{1}{0.301}\left[0.5\log\frac{2}{2} + 0.242\log\frac{8}{5} + 0.258\log\frac{14}{6.548}\right]$$

$\approx$3.32[(0,242).(0.204)+(0.258).(0.33)] $\approx$0.445 and

$$\text{N}(r) = \frac{1}{\log 2}\left[\sum_{i=2}^{n}(r_i - r_{i+1})\log i\right] == \frac{1}{\log 2}\,[0.5$$

$$\log 2 + 0.242\log 8 + 0.258\log 14]$$

$\approx$0.5+3.(0.242)+(0.857).1.146$\approx$2.208 . Therefore we finally obtained that T(r)$\approx$2.653.

A few days later we performed the same experiment with a group of 30 students of the School of Management and Economics. Working as above we found that

A_1=*{(a,0),(b, 0.25),(c, 0.5),(d, 0.25),(e,0)}*,

A_2=*{(a, 0.25),(b, 0.25),(c, 0.5),(d, 0),(e,0)}* and

A_3=*{(a, 0.25),(b, 0.25),(c,0.25),(d,0),(e,0)}*.

Then we calculated the membership degrees of all possible profiles of the student group (see column of m_s (2) in Table 1). It turned out that the maximal membership degree was again 0.06225, therefore the possibility of each s is given by the same formula as for the first group. Calculating the possibilities of all profiles (see column of $r_s(2)$ in Table 1) we found that the ordered possibility distribution of the second group is:

$r_1=r_2=1$, $r_3=r_4=r_5=r_6=r_7=r_8=0{,}5$, $r_9=r_{10}=r_{11}=r_{12}=r_{13}=0{,}258$,
$r_{14}=r_{15}=\ldots\ldots=r_{125}=0$

Finally, working in the same way as above we found that T*(r) = 0.432+2.179 = 2.611*. Therefore, since 2.611<2.653, it turns out that the second group had in general a slightly better performance than the first one.

Next, in order to study the combined results of behaviours of the two groups, we introduced the fuzzy variables A_i (t), i=1, 2, 3 and t=1, 2, as we have described in the previous section. Then the pseudo-frequency of each student profile s is given by $f(s) = m_s(1) + m_s(2)$ (see corresponding column in Table 1). It turns out that the highest pseudo-frequency is 0.124 and therefore the possibility of each student's profile is given by $r(s) = \frac{f(s)}{0.124}$. The possibilities of all profiles having non-zero pseudo-frequencies are presented in the last column of Table 1.

Discussion and conclusions

In this paper we developed a fuzzy model for representing the P-S process. and we used the total possibilistic uncertainty as a measure of students'

P-S capacities. We also presented two classroom experiments illustrating our results in practice. Nevertheless further research is needed for the P-S process. In fact, as a general conclusion of all findings from research studies on P-S it turns out that success in P-S appears to stem from the ability to draw on a large reservoir of well-connected knowledge, heuristics and facts, from the ability to manage the emotional responses, as well as from an adequate degree of practice (Voskoglou 2007a, 2008). However, although many studies have investigated and compared the characteristics of novice and expert problem solvers (Lesh and Akerstrom 1982, Schoenfeld 1985a, Stillman and Galbraith 1998, etc), many of the qualitative differences appearing among them still do not seem to be completely understood. It is hopped therefore that the use of our fuzzy model as a tool in future research on P-S could lead to practical ways of restoring the weaknesses appearing to novices with respect to the expert problem solvers.

Similar models were used in earlier papers for an effective description of situations involving fuzziness and uncertainty, mainly in the area of Mathematics Education (Learning, Mathematical Modelling), but also in the areas of Artificial Intelligence (Case-Based Reasoning) and Management. All these models were developed by adapting properly the general fuzzy model presented in this paper. It is hopped to be able in future to extend our general model in representing further situations involving fuzziness and uncertainty either in Education and/or in other scientific areas.

Our fuzzy models, apart from quantitative information (e.g. possibilities, total possibilistic uncertainty of the system, etc), they also provide a realistic qualitative view of the process that they represent through the study of all possible profiles during the process of the subjects involved. Another advantage of them is that they give the opportunity for a combined study of results of two or more groups (or systems) during the same situation, or alternatively for a combined study of results of the same group (or system) during two or more different situations. Notice that analogous efforts to use principles of fuzzy logic in Education have been attempted in past by other researchers as well (Espin and Oliveras 1997, Ma and Zhou 2000, Perdikaris 2011, Spagnolo 2003, Subbotin et al. 2006, etc).

We must finally underline the importance of use of *stochastic (Markov chain) models* as an alternative approach for the same purposes; e.g. Voskoglou and Perdikaris 1991, Voskoglou 1996, 2000, 2007b, 2009b, 2009d, 2010a, etc. These models provide also useful quantitative information like measures for the P-S or model-building abilities of student groups, short and long-run forecasts (probabilities) for the evolution of various phenomena, etc. Nevertheless they are self restricted in describing the *ideal behaviour* only of the subjects involved , in which they proceed linearly through the several stages of the corresponding process arriving to acceptable solutions and reporting on them. Therefore one could claim that the fuzzy models are more useful for a deeper study of the corresponding real situations, because they provide also the possibility of a realistic qualitative analysis of the problems involved.

Table 1
Profiles with non zero pseudo-frequencies

A_1	A_2	A_3	$m_s(1)$	$r_s(1)$	$m_s(2)$	$r_s(2)$	$f(s)$	$r(s)$
b	b	b	0	0	0.016	0.258	0.016	0.129
b	b	a	0	0	0.016	0.258	0.016	0.129
b	a	a	0	0	0.016	0.258	0.016	0.129
c	c	c	0.062	1	0.062	1	0.124	1
c	c	a	0.062	1	0.062	1	0.124	1
c	c	b	0	0	0.031	0.5	0.031	0.25
c	a	a	0	0	0,031	0.5	0.031	0.25
c	b	a	0	0	0.031	0.5	0.031	0.25
c	b	b	0	0	0.031	0.5	0.031	0.25
d	d	a	0.016	0.258	0	0	0.016	0.129
d	d	b	0.016	0.258	0	0	0.016	0.129
d	d	c	0.016	0.258	0	0	0.016	0.129
d	a	a	0	0	0.016	0.258	0.016	0.129
d	b	a	0	0	0.016	0.258	0.016	0.129
d	b	b	0	0	0.016	0.258	0.016	0.129
d	c	a	0.031	0.5	0.031	0.5	0.062	0.5
d	c	b	0.031	0.5	0.031	0.5	0.062	0.5
d	c	c	0.031	0.5	0.031	0.5	0.062	0.5
e	c	a	0.031	0.5	0	0	0.031	0.25
e	c	b	0.031	0.5	0	0	0.031	0.25
e	c	c	0.031	0.5	0	0	0.031	0.25
e	d	a	0.016	0.258	0	0	0.016	0.129
e	d	b	0.016	0.258	0	0	0.016	0.129
e	d	c	0.016	0.258	0	0	0.016	0.129

(The outcomes of the table are written with accuracy up to the third decimal point)

APPENDIX
List of the problems given to students for solution in our classroom experiments

Problem 1: We want to construct a channel to run water by folding across its longer side the two edges of an orthogonal metallic leaf having sides of length 20cm and 32 cm, in such a way that they will be perpendicular to the other parts of the leaf. Assuming that the flow of the water is constant, how we can run the maximum possible quantity of the water?

Problem 2: Given the matrix A= $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ and a positive integer n, find the matrix A^n.

Problem 3: Calculate the integral $\int \frac{x}{x^2+4}dx$.

Problem 4: Let us correspond to each letter the number showing its order into the alphabet (A=1, B=2, C=3 etc). Let us correspond also to each word consisting of 4 letters a 2X2 matrix in the obvious way; e.g. the matrix $\begin{bmatrix} 19 & 15 \\ 13 & 5 \end{bmatrix}$ corresponds to the word SOME. Using the matrix E= $\begin{bmatrix} 8 & 5 \\ 11 & 7 \end{bmatrix}$ as an encoding matrix how you could send the message LATE in the form of a camouflaged matrix to a receiver knowing the above process and how he (she) could decode your message?

Problem 5: The demand function $P(Q_d)=25-Q_d^2$ represents the different prices that consumers willing to pay for different quantities Q_d of a good. On the other hand the supply function $P(Q_s)=2Q_s+1$ represents the prices at which different quantities Q_s of the same good will be supplied. If the market's equilibrium occurs at (Q_0, P_0), the producers who would supply at lower price than P_0 benefit. Find the total gain to producers'.

Problem 6: A ballot box contains 8 balls numbered from 1 to 8. One makes 3 successive drawings of a lottery, putting back the corresponding ball to the box before the next lottery. Find the probability of getting all the balls that he draws out of the box different.

Problem 7: A box contains 3 white, 4 blue and 6 black balls. If we put out 2 balls, what is the probability of choosing 2 balls of the same colour?

Problem 8: The rate of increase of the population of a country is analogous to the number of its inhabitants. If the population is doubled in 50 years, in how many years it will be tripled? (ANSWER: In $50\frac{\ln 3}{\ln 2} \approx 79$ years).

Problem 9: A company circulates for first time in market a new product, say K. Market's research has shown that the consumers buy on average one such product per week, either K, or a competitive one. It is also expected that 70% of those who buy K they will prefer it again next week, while 20% of those who buy another competitive product they will turn to K next week.

i) Find the market's share for K two weeks after its first circulation, provided that the market's conditions remain unchanged.

ii) Find the market's share for K in the long run, i.e. when the consumers' preferences will be stabilized.

Problem 10: Among all cylinders having a total surface of $180\pi\ m^2$, which one has the maximal volume

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References

Carlson, M. P. & Bloom, I. (2005), The cyclic nature of problem solving: An emergent multidimensional problem solving framework, Educational Studies in Mathematics, 58, 45-75.

Espin, E. A. & Oliveras, C. M. L. (1997), Introduction to the use of fuzzy logic in the assessment of mathematics teachers', in: A. Gagatsis (Ed.), Proceedings of the 1st Mediterranean Conference on Mathematics Education, 107-113, Nicosia, Cyprus, 107-113.

Klir, G. J. & Folger, T. A. (1988), Fuzzy Sets, Uncertainty and Information, Prentice-Hall, London, 1988.

Klir, G. J. (1995), Principles of uncertainty: What are they? Why do we need them?, Fuzzy Sets and Systems, 74, 15-31.

Lesh, R. & Akerstrom, M. (1982), Applied problem solving: Priorities for mathematics education research, in: F. K. Lester & J. Garofalo (Eds.), Mathematical Problem Solving: Issues in Research, Franklin Institute Press, Philadelphia, 117-129.

Lester, F. K.& Garofalo, J. & Krol, D. L. (1989), Self-confidence, interest, beliefs and metacognition: Key influences on problem-solving behavior", in: Mcleod, D. B. & Adams, V. M. (Eds), Affect and Mathematical Problem Solving: A New Perspective, Springer-Verlag, New York, 75-88.

Ma, J. & Zhou, D. (2000), "Fuzzy Set Approach to the Assessment of Student-Centered Learning", IEEE Transactions on Education, 43(2), 237-241.

Perdikaris, S. C, (2011) Using Fuzzy Sets to Determine the Continuity of the van Hiele Levels, Journal of Mathematical Sciences & Mathematics Education, 5(1), 39-46.

Polya, G. (1945), How to solve it, Princeton Univ. Press, Princeton.

Polya, G. (1954), Mathematics and Plausible Reasoning, Princeton Univ. Press, Princeton.

Polya, G. (1963), On learning, teaching and learning teaching, American Mathematical Monthly, 70, 605-619.

Polya, G. (1962/65), Mathematical Discovery (2 Volumes), J.Wilet $ Sons, New York.

Shackle, G. L. S. (1961), Decision, Order and Time in Human Affairs, Cambridge University Press, Cambridge.

Schoenfeld, A. (1980), Teaching problem solving skills, American Mathematical Monthly, 87, 794-805.

Schoenfeld, A. (1985a), Problem solving in context(s), in: E. A. Silver (Ed.), The teaching and Assessing of Mathematical Problem-Solving, Vol. 3, Reston, VA.

Schoenfeld, A. (1985b), Mathematical problem solving, Academic Press Orlando, FL.

Schoenfeld , A.(2007), Problem solving, teaching, and more: Toward a theory of goal-directed behaviour, Proceedings CIEAEM 59, 48-52, Dobogoko, Hungary.

Spagnolo, F. (2003), Fuzzy logic, fuzzy thinking and the teaching/ learning of mathematics in multicultural situations, Proceedings International Conference on Mathematics Education into the 21st Century (MEC21), Brno (Czech Republic), 17-28.

Stillman, G. A. & Galbraith, P. (1998), Applying mathematics with real world connections: Metacognitive characteristics of secondary students, Educational Studies in Mathematics, 96 , 157-189.

I. Subbotin, Ya. H. Badkoodehi, H. & Bilotskii, N. N. (2006), Fuzzy logic and iterative assessment, Didactics of Mathematics: Problems and Investigations (Ukraine), 25, 221-227.

Voskoglou, M. Gr. & Perdikaris, S. (1991) , A Markov chain model in problem solving, International Journal of Mathematical Education in Science and Technology, 22(6), 909-914.

Voskoglou, M. Gr. (1996), An application of ergodic Markov chains to analogical problem solving, The Mathematics Education (India), Vol. XXX, No 2, 95-108.

Voskoglou, M. Gr. (2000), An application of Markov chains to decision making", *Studia Kupieckie* (Lodz-Poland), 6 (2000), 69-76.

Voskoglou, M. Gr. (2003), Applications of Fuzzy Sets to problems of commercial enterprises, Proceedings 1st International Conference on Quantitative Methods in Industry and Commerce, T. E. I. of Athens, 654-659.

Voskoglou, M. Gr. (2006), The use of mathematical modelling as a learning device of mathematics, Quaderni di Ricerca in Didattica (Scienze Mathematiche), University of Palermo, 16 , 53-60.

Voskoglou, M. Gr. (2007a), The problem as a learning device of mathematics, in P. Avgerinos & A. Gagatsis (Eds.), Current Trends in Mathematics Education, New Technologies Publications, Athens, 223-232.

Voskoglou, M. Gr. (2007b), A stochastic model for the modelling process, in: C. Chaines, P. Galbraith, W. Blum & S. Khan (Eds.) Mathematical Modelling: Education, Engineering and Economics (ICTMA 12), Horwood Publishing, Chichester, 149-157.

Voskoglou, M. Gr. (2007c), The role of the teacher for the learning of mathematics, Proceedings CIEAEM 59, Dobogoko, Hungary, 2007, 278-283.

Voskoglou, M. Gr., Problem- Solving in Mathematics Education: Recent Trends and Development (2008), Quaderni di Ricerca in Didattica (Scienze Mathematiche), University of Palermo, 18, 22-28.

Voskoglou, M. Gr. (2009a), Fuzziness or probability in the process of learning: A general question illustrated by examples from teaching mathematics, The Journal of Fuzzy Mathematics, International Fuzzy Mathematics Institute (Los Angeles), 17(3), 679-686.

Voskoglou, M. Gr. (2009b), Applications of Markov chains to business problems, Proceedings 2nd International Conference on Quantitative and Qualitative Methodologies in the Economic and Administrative Sciences, T. E. I. of Athens, 528-535.

Voskoglou, M. Gr. (2009c), Fuzzy sets in Case-Based Reasoning, Fuzzy Systems and Knowledge Discovery (FSKD 2009), IEEE Computer Society, 6252-256.

Voskoglou, M. Gr. (2009d), A stochastic model for the process of learning, in: L. Paditz & A. Rogerson (Eds.), Models in Developing Mathematics Education, University of Applied Sciences, Dresden, 565-569.

Voskoglou, M. Gr. 2010a),A stochastic model for Case-Based Reasoning, International Journal of Modelling and Application, University Blumenau, Brazil, 3, 33- 39.

Voskoglou, M. Gr. (2010b), Use of total possibilistic uncertainty as a measure of students' modelling capacities, International Journal of Mathematical Education in Science and Technology, 41(8), 1051-1060.

Voskoglou, M. Gr. (2011), Measuring students' modelling capacities: A fuzzy approach, Iranian Journal of Fuzzy Systems, 8(3), to appear.