On A General Formula of Fourth Order Runge-Kutta Method

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Abstract

In this paper, we obtain a general formula of Runge-Kutta method in order 4 with a free parameter *t*. By picking the value of *t*, it can generate many RK methods in order 4 including some known results. Numerical calculation shows that those new formulas have the same accuracy as the classical RK4 has.

I. Introduction and Main Results

The Runge-Kutta algorithm is used for solving the numerical solution of the ordinary differential equation $y'(x) = f(x, y)$ with initial condition $y(x_0) = w_0$. In about 1900, Runge and Kutta developed the following classical fourth order Runge-Kutta iterative method

$$(1)\quad \begin{cases} w_{k+1} = w_k + \frac{1}{6}\left(K_1 + 2K_2 + 2K_3 + K_4\right)h, \\ K_1 = f(x_k,\ w_k), & K_2 = f\left(x_k + \frac{1}{2}h,\ w_k + \frac{1}{2}K_1 h\right), \\ K_3 = f\left(x_k + \frac{1}{2}h,\ w_k + \frac{1}{2}K_2 h\right), & K_4 = f\left(x_k + h,\ w_k + K_3 h\right). \end{cases}$$

which has the accumulative error in order of $O(h^4)$.

Since then, many mathematicians have tried to develop more Runge-Kutta like methods in a variety of directions. In 1969, R. England [1] developed another fourth order Runge-Kutta method (See the book of L. F. Shampine, R. C. Allen and S. Pruess [2])

$$(2)\quad \begin{cases} w_{k+1} = w_k + \frac{1}{6}\left(K_1 + 4K_3 + K_4\right)h, \\ K_1 = f(x_k,\ w_k), & K_2 = f\left(x_k + \frac{1}{2}h,\ w_k + \frac{1}{2}K_1 h\right), \\ K_3 = f\left(x_k + \frac{1}{2}h,\ w_k + \frac{1}{4}K_1 h + \frac{1}{4}K_2 h\right), & K_4 = f\left(x_k + h,\ w_k - K_2 h + 2K_3 h\right). \end{cases}$$

Although the higher order of Runge-Kutta methods can increase the error accuracy, but it would need to pay the cost of calculation complexity. Therefore the most mathematicians think that the fourth order Runge-Kutta method is the most efficient method to solve numerical solution for IVP. It is naturally to ask: Do there exist other fourth order Runge-Kutta methods? If yes, then what are they look like? In this paper, we obtain a general formula fourth order Runge-Kutta method with a free parameter as the following

$$
(3)\quad \begin{cases}
w_{k+1} = w_k + \frac{1}{6}\left[K_1 + (4-t)K_2 + tK_3 + K_4\right]h, \\
K_1 = f(x_k,\ w_k), \\
K_2 = f\left(x_k + \frac{1}{2}h,\ w_k + \frac{1}{2}K_1 h\right), \\
K_3 = f\left(x_k + \frac{1}{2}h,\ w_k + \left(\frac{1}{2} - \frac{1}{t}\right)K_1 h + \frac{1}{t}K_2 h\right), \\
K_4 = f\left(x_k + h,\ w_k + \left(1 - \frac{t}{2}\right)K_2 h + \frac{t}{2}K_3 h\right),
\end{cases}
$$

where t is a free parameter.

Remark 1: Pick $t = 2$, then (3) becomes the classical Runge-Kutta method (1).
Remark 2: Pick $t = 4$, then (3) becomes England's Runge-Kutta method (2).
Remark 3: Pick $t = 1$, we get a new formula for Runge-Kutta method

$$
(4)\quad \begin{cases}
w_{k+1} = w_k + \frac{1}{6}(K_1 + 3K_2 + K_3 + K_4)h, \\
K_1 = f(x_k,\ w_k), \\
K_2 = f\left(x_k + \frac{1}{2}h,\ w_k + \frac{1}{2}K_1 h\right), \\
K_3 = f\left(x_k + \frac{1}{2}h,\ w_k - \frac{1}{2}K_1 h + K_2 h\right), \\
K_4 = f\left(x_k + h,\ w_k + \frac{1}{2}K_2 h + \frac{1}{2}K_3 h\right).
\end{cases}
$$

Remark 4: Pick $t = 3$ we get the second new formula for Runge-Kutta method

$$
(5)\quad \begin{cases}
w_{k+1} = w_k + \frac{1}{6}(K_1 + K_2 + 3K_3 + K_4)h, \\
K_1 = f(x_k,\ w_k), \\
K_2 = f\left(x_k + \frac{1}{2}h,\ w_k + \frac{1}{2}K_1 h\right), \\
K_3 = f\left(x_k + \frac{1}{2}h,\ w_k + \frac{1}{6}K_1 h + \frac{1}{3}K_2 h\right), \\
K_4 = f\left(x_k + h,\ w_k - \frac{1}{2}K_2 h + \frac{3}{2}K_3 h\right).
\end{cases}
$$

Remark 5: Pick $t = 5$, we get the third new formula for Runge-Kutta method

$$
(6)\quad \begin{cases}
w_{k+1} = w_k + \frac{1}{6}(K_1 - K_2 + 5K_3 + K_4)h, \\
K_1 = f(x_k,\ w_k), \\
K_2 = f\left(x_k + \frac{1}{2}h,\ w_k + \frac{1}{2}K_1 h\right), \\
K_3 = f\left(x_k + \frac{1}{2}h,\ w_k + \frac{3}{10}K_1 h + \frac{1}{5}K_2 h\right), \\
K_4 = f\left(x_k + h,\ w_k - \frac{3}{2}K_2 h + \frac{5}{2}K_3 h\right).
\end{cases}
$$

II. Numerical Computing Tests

With the same step value of *h*, we use the different fourth order Runge-Kutta methods, including the classical one, to calculate the numerical solution for several differential equations with initial value condition at $x = 0$. Then comparing to the real analytic solutions to estimate the relative error at $x = 4$.
Our first example is to calculate the numerical solution for

$$y' = y - x^2 + 1 \quad (0 \le x \le 4) \text{ with } y(0) = 3 \text{ and } h = 0.2.$$

Comparing to the analytic solution $y = 2e^x + x^2 + 2x + 1$ at $x = 4$, we obtain the following results:

RK4 methods	Classical	R. England	Formula (4)	Formula (5)	Formula (6)
Relative errors at x=4	0.0049%	0.0049%	0.0049%	0.0049%	0.0049%

Our second example is to calculate the numerical solution for

$$y' = xy - x^3 \qquad (0 \le x \le 4) \text{ with } y(0) = 3 \text{ and } h = 0.2.$$

Comparing to the analytic solution $y = e^{x^2/2} + x^2 + 2$, we obtain the following results:

RK4 methods	Classical	R. England	Formula (4)	Formula (5)	Formula (6)
Relative errors at x=4	0.0053%	0.0053%	0.0053%	0.0053%	0.0053%

Our last example is to calculate the numerical solution for the second order differential equation

$$y'' = y + 2\sin x \ (0 \le x \le 4) \text{ with } y(0) = 1,\ y'(0) = 0 \text{ and } h = 0.2.$$

Define vector variable $Y = \begin{bmatrix} y \\ y' \end{bmatrix}$ and apply the different fourth order Runge-Kutta methods to the following vector differential equation

$$\frac{dY}{dx} = f(x, Y) \text{ with initial condition } Y(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\text{where } f(x, Y) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} Y + \begin{bmatrix} 0 \\ 2\sin x \end{bmatrix}$$

After comparing to the analytic solution $y = e^x - \sin x$, we get the following results

RK4 methods	Classical	R. England	Formula (4)	Formula (5)	Formula (6)
Relative errors at x=4	0.0051%	0.0051%	0.0051%	0.0051%	0.0051%

The above calculating results show that the new fourth order Runge-Kutta methods formulas (4) (5) and (6) have the exactly same error accuracy as the classical method (1) does.

III. Lemma

To derive the above general formula (3), we need the following lemma:

Lemma: Suppose that a, b, c, d, p_1, p_2, p_3, q_1, q_{2a}, q_{2b}, q_{3a} and q_{3b} are real numbers and function $f(x, y)$ is continues in a neighborhood of a point (x_0, y_0). Let $y = y(x)$ be a solution of differential equation $y'(x) = f(x, y)$ with initial condition $y(x_0) = y_0$. Define

(7) $$\begin{cases} K_1 = f(x, y), \\ K_2 = f(x + p_1 h, y + q_1 K_1 h), \\ K_3 = f(x + p_2 h, y + q_{2a} K_1 h + q_{2b} K_2 h), \\ K_4 = f(x + p_3 h, y + q_{3a} K_2 h + q_{3b} K_3 h) \end{cases}$$

For small $h > 0$, if

(8) $$y(x+h) = y(x) + (aK_1 + bK_2 + cK_3 + dK_4)h + O(h^5)$$

is true for any continuous function $f(x, y)$ in the domain of (x, y), then

(9) $$\begin{cases} a + b + c + d = 1, \\ p_1 = q_1, \\ p_2 = q_{2a} + q_{2a}, \\ p_3 = q_{3a} + q_{3b} \\ 2(bp_1 + cp_2 + dp_3) = 1, \\ 3(bp_1^2 + cp_2^2 + dp_3^2) = 1, \\ 4(bp_1^3 + cp_2^3 + dp_3^3) = 1, \\ 6\left(cp_1 q_{2b} + dp_1 q_{3a} + dp_2 q_{3b}\right) = 1, \\ 24 dp_1 q_{2b} q_{3b} = 1 \\ 4\left(p_1 + p_2 + 6dp_1 p_3 q_{3a} + 6dp_2 p_3 q_{3b} - 6dp_1 p_2 p_3\right) = 5. \end{cases}$$

Proof: First pick $y(x) = x$, then $f(x, y) = y'(x) = 1$ and $K_1 = K_2 = K_3 = K_4 = 1$. Plug them into (8) we get

$$x + h = x + (a + b + c + d)h + O(h^5).$$

Therefore we obtain

(11) $$a + b + c + d = 1.$$

Still consider , but setup $f(x, y) = y'(x) = \frac{y}{x}$. Then

$$K_1 = \frac{y}{x} = 1, \qquad K_2 = \frac{y + q_1 K_1 h}{x + p_1 h} = \frac{x + q_1 h}{x + p_1 h},$$

$$K_3 = \frac{x + q_{2a} K_1 h + q_{2b} K_2 h}{x + p_2 h}, \qquad K_4 = \frac{x + q_{3a} K_2 h + q_{3b} K_3 h}{x + p_3 h}.$$

To make the point (x_k, w_k) on the solution line $y = x$, K_2 should be equal one. Therefore, the only possible relation between p_1 and q_1 is

(12) $$p_1 = q_1.$$

Then $$K_3 = \frac{x + \left(q_{2a} + q_{2b}\right)h}{x + p_2 h}$$

For the same arguments on K_3, we obtain

(13) $$p_2 = q_{2a} + q_{2b}.$$

Then $$K_4 = \frac{x + \left(q_{3a} + q_{3b}\right)h}{x + p_3 h}$$

Similarly, we obtain

(14) $$p_3 = q_{3a} + q_{3b}.$$

If pick $y(x) = x^2$ and setup $f(x, y) = y'(x) = 2x$, then

$$K_1 = 2x,\ K_2 = 2(x + p_1 h),\ K_3 = 2(x + p_2 h),\ K_4 = 2(x + p_3 h).$$

Plug into (8)

$$(x+h)^2 = x^2 + h\left[2ax + 2b(x + p_1 h) + 2c(x + p_2 h) + 2d(x + p_3 h)\right] + O(h^5)$$
$$= x^2 + 2(a+b+c+d)xh + 2\left(bp_1 + cp_2 + dp_3\right)h^2 + O(h^5).$$

Then we get

(15) $$2\left(bp_1 + cp_2 + dp_3\right) = 1.$$

Next pick $y(x) = x^3$ and setup $f(x, y) = y'(x) = 3x^2$, then

$$K_1 = 3x^2,\ K_2 = 3(x + p_1 h)^2,\ K_3 = 3(x + p_2 h)^2,\ K_4 = 3(x + p_3 h)^2.$$

Plug into (8)

$$(x+h)^3 = x^3 + h\left[3ax^2 + 3b(x + p_1 h)^2 + 3c(x + p_2 h)^2 + 3d(x + p_3 h)^2\right] + O(h^5)$$
$$= x^3 + 3x^2 h + 3xh^2 + 3\left(bp_1^2 + cp_2^2 + dp_3^2\right)h^3 + O(h^5)$$

Then we get

(16) $$3\left(bp_1^2 + cp_2^2 + dp_3^2\right) = 1$$

If pick $y(x) = x^4$ and setup $f(x, y) = y'(x) = 4x^3$, then

$$K_1 = 4x^3,\ K_2 = 4(x + p_1 h)^3,\ K_3 = 4(x + p_2 h)^3,\ K_4 = 4(x + p_3 h)^3.$$

Plug into (8)

$$(x+h)^4 = x^4 + h\left[4ax^3 + 4b(x + p_1 h)^3 + 4c(x + p_2 h)^3 + 4d(x + p_3 h)^3\right] + O(h^5)$$
$$= x^4 + 4x^3 h + 6x^2 h^2 + 4xh^3 + 4\left(bp_1^3 + cp_2^3 + dp_3^3\right)h^4 + O(h^5).$$

Then we get

(17) $$4\left(bp_1^3 + cp_2^3 + dp_3^3\right) = 1.$$

Again pick $y(x) = x^3$ but setup $f(x, y) = y'(x) = \dfrac{3y}{x}$. Then $K_1 = 3x^2$ and

$$K_2 = \frac{3(y+q_1K_1h)}{x+p_1h} = \frac{3(x^3+3p_1x^2h)}{x+p_1h}$$

$$= 3\left[x^2+2p_1xh-2(p_1h)^2\right]+O(h^3)$$

$$K_3 = \frac{3(y+q_{2a}K_1h+q_{2b}K_2h)}{x+p_2h}$$

$$= \frac{3(x^3+3q_{2a}x^2h+3q_{2b}x^2h+6p_1q_{2b}xh^2)}{x+p_2h} + O(h^3)$$

$$= \frac{3(x^3+3p_2x^2h+6p_1q_{2b}xh^2)}{x+p_2h}+O(h^3)$$

$$= 3\left[x^2+2p_2xh-2(p_2h)^2+6p_1q_{2b}h^2\right]+O(h^3)$$

$$K_4 = \frac{3(y+q_{3a}K_2h+q_{3b}K_3h)}{x+p_3h}$$

$$= \frac{3(x^3+3q_{3a}x^2h+6p_1q_{3a}xh^2+3q_{3b}x^2h+6p_2q_{3b}xh^2)}{x+p_3h}+O(h^3)$$

$$= \frac{3\left[x^3+3p_3x^2h+6(p_1q_{3a}+p_2q_{3b})xh^2\right]}{x+p_3h}+O(h^3)$$

$$= 3\left[x^2+2p_3xh-2x(p_3h)^2+6(p_1q_{3a}+p_2q_{3b})h^2\right]+O(h^3)$$

Plug into (8)

$$\begin{aligned}(x+h)^3 = x^3 &+ 3ax^2h+3b\left[x^2+2p_1xh-2(p_1h)^2\right]h \\ &+3c\left[x^2+2p_2xh-2(p_2h)^2+6p_1q_{2b}h^2\right]h \\ &+3d\left[x^2+2p_3xh-2x(p_3h)^2+6(p_1q_{3a}+p_2q_{3b})h^2\right]h+O(h^4) \\ &= x^2+3(a+b+c+d)x^2h+6(bp_1+cp_2+dp_3)xh^2 \\ &+3\left[-2(bp_1^2+cp_2^2+dp_3^2)+6(dp_1q_{3a}+dp_2q_{3b}+cp_1q_{2b})\right]h^3+O(h^4).\end{aligned}$$

Then we get

$$3\left[-2(bp_1^2+cp_2^2+dp_3^2)+6(dp_1q_{3a}+dp_2q_{3b}+cp_1q_{2b})\right]=1$$

From (17)

$$18(dp_1q_{3a}+dp_2q_{3b}+cp_1q_{2b})=6(bp_1^2+cp_2^2+dp_3^2)+1=3$$

Therefore

(18) $$6(cp_1q_{2b}+dp_1q_{3a}+dp_2q_{3b})=1.$$

Still consider $y(x)=x^4$ and setup $f(x,y)=y'(x)=\dfrac{4y}{x}$, then $K_1=4x^3$ and

$$K_2=\frac{4\left(y+q_1K_1h\right)}{x+p_1h}$$

$$K_3=\frac{4\left(y+q_{2a}K_1h+q_{2b}K_2h\right)}{x+p_2h}$$

$$=\frac{4\left[x^4+4q_{2a}x^3h+q_{2b}\left(4x^3+12p_1x^2h-12\left(p_1h\right)^2x\right)h\right]}{x+p_2h}+O(h^4)$$

$$=\frac{4\left[x^4+4p_2x^3h+12p_1q_{2b}x^2h^2-12p_1^2q_{2b}xh^3\right]}{x+p_2h}+O(h^4)$$

$$=4\left[x^3+3\left(p_2h\right)x^2-3\left(p_2h\right)^2x+3\left(p_2h\right)^3\right]$$
$$+4\left[12p_1q_{2b}xh^2-12p_1p_2q_{2b}h^3-12p_1^2q_{2b}h^3\right]+O(h^4).$$

Because

$$y+q_{3a}K_2h+q_{3b}K_3h$$
$$=x^4+4q_{3a}\left[x^3+3\left(p_1h\right)x^2-3\left(p_1h\right)^2x\right]h+O(h^4)$$
$$+4q_{3b}\left[x^3+3\left(p_2h\right)x^2-3\left(p_2h\right)^2x+12p_1q_{2b}xh^2\right]h+O(h^4)$$
$$=x^4+4q_3x^3h+12p_1q_{3a}x^2h^2-12p_1^2q_{3a}xh^3+4q_4x^3h$$
$$+12p_2q_{3b}x^2h^2-12p_2^2q_{3b}xh^3+48p_1q_{2b}q_{3b}xh^3+O(h^4)$$
$$=x^4+4p_3x^3h+12\left(p_1q_{3a}+p_2q_{3b}\right)x^2h^2-12\left(p_1^2q_{3a}+p_2^2q_{3b}\right)xh^3$$
$$+48p_1q_{2b}q_{3b}xh^3+O(h^4).$$

Then

$$K_4=\frac{4\left(y+q_{3a}K_2h+q_{3b}K_3h\right)}{x+p_3h}$$

$$=\frac{4\left[x^4+4p_3x^3h+12\left(p_1q_{3a}+p_2q_{3b}\right)x^2h^2-12\left(p_1^2q_{3a}+p_2^2q_{3b}\right)xh^3+48p_1q_{2b}q_{3b}xh^3\right]}{x+p_3h}+O(h^4)$$

$$=4\left[x^3+3\left(p_3h\right)x^2-3\left(p_3h\right)^2x+3\left(p_3h\right)^3+12\left(p_1q_{3a}+p_2q_{3b}\right)xh^2-12\left(p_1q_{3a}+p_2q_{3b}\right)p_3h^3\right]$$
$$+4\left[-12\left(p_1^2q_{3a}+p_2^2q_{3b}\right)h^3+48p_1q_{2b}q_{3b}h^3\right]+O(h^4).$$

Plug into (8)

$$(x+h)^4 = x^4 + 4ax^3h + 4b\left[x^3 + 3p_1x^2h - 3(p_1h)^2 x + 3(p_1h)^3\right]h$$
$$+4c\left[x^3 + 3(p_2h)x^2 - 3(p_2h)^2 x + 3(p_2h)^3 + 12p_1q_{2b}xh^2 - 12p_1p_2q_{2b}h^3 - 12p_1^2q_{2b}h^3\right]h$$
$$+4d\left[x^3 + 3(p_3h)x^2 - 3(p_3h)^2 x + 3(p_3h)^3 + 12(p_1q_{3a} + p_2q_{3b})xh^2\right]h$$
$$+4d\left[-12(p_1q_{3a} + p_2q_{3b})p_3h^3 - 12(p_1^2q_{3a} + p_2^2q_{3b})h^3 + 48p_1q_{2b}q_{3b}h^3\right]h + O(h^5)$$

$$= x^4 + 4(a+b+c+d)x^3h + 12(bp_1 + cp_2 + dp_3)x^2h^2$$
$$+4\left[-3(bp_1^2 + cp_2^2 + dp_2^2) + 12(cp_1q_{2b} + dp_1q_{3a} + dp_2q_{3b})\right]xh^3 + 12\left[bp_1^3 + cp_2^3 + dp_3^3\right]h^4$$
$$+4\left[-12p_1(cp_1q_{2b} + dp_1q_{3a} + dp_2q_{3b}) - 12p_2(cp_1q_{2b} + dp_1q_{3a} + dp_2q_{3b})\right]h^4$$
$$+4\left[-12dp_1p_3q_{3a} - 12dp_2p_3q_{3b} + 12dp_1p_2p_3 + 48dp_1p_2q_{3b}\right]h^4 + O(h^5)$$
$$= x^4 + 4x^3h + 6x^2h^2 + 4xh^3h^4 + 3h^4 + 4(-2p_1 - 2p_2)h^4$$
$$+4\left[-12dp_1p_3q_{3a} - 12dp_2p_3q_{3b} + 12dp_1p_2p_3 + 48dp_1q_{2b}q_{3b}\right]h^4 + O(h^5)$$

Therefore

$$3 + 4(-2p_1 - 2p_2 + 12dp_1p_2q_{3a} - 12dp_1p_3q_{3a} - 12dp_2p_3q_{3b} + 48dp_1q_{2b}q_{3b}) = 1$$

We obtain

(18)

$$4(p_1 + p_2 + 6dp_1p_3q_{3a} + 6dp_2p_3q_{3b} - 6dp_1p_2p_3 - 24dp_1q_{2b}q_{3b}) = 1$$

Thus, we have obtained all equations in (9) of the lemma.

Remark 1: Equations in lemma are the necessary conditions to make the RK4 constants a, b, c, d, p_1, p_2, p_3, q_1, q_{2a}, q_{2b}, q_{3a} and q_{3b}. They might not be enough make (8) to be true for some $f(x,y)$.

Remark 2: Equations in (9) plus the following equations

(21)
$$\begin{cases} 24dp_1q_{2b}q_{3b} = 1 \\ 6(cp_1q_{2b} + dp_1q_{3a} + dp_2q_{3b}) = 1, \\ 8(cp_1p_2q_{2b} + dp_1p_3q_{3a} + dp_2p_3q_{3b}) = 1 \end{cases}$$

would be enough to make (8) to be true for all $f(x,y)$. See the book of J. H. Mathews and K. D. Fink [3].

IV. Derivation the General RK4 formulas

The total number of questions in systems (9) and (21) is 12. The number of all unknown constants is also 12. Therefore, it could have only one solution for constants a, b, c, d, p_1, p_2, p_3, q_1, q_{2a}, q_{2b}, q_{3a} and q_{3b} theoretically. How if we set up $p_1 = p_2$ or $p_1 = p_3$ or $p_1 = p_3$, then the rank of the following equation system

$$\begin{cases} 2(bp_1+cp_2+dp_3)=1 \\ 3(bp_1^2+cp_2^2+dp_3^2)=1 \\ 4(bp_1^3+cp_2^3+dp_3^3)=1 \end{cases}$$

Would be reduced to two, then we would get a free parameter. By a tedious analyzing, we have found that the only possible case is $p_1= p_2$. The further calculation reveals that this common value is only possibly to be 1/2 or 1/3. If $p_1 = p_2 = 1/2$, we can calculate to get $p_3=1$. If $p_1= p_2 = 1/3$, we can calculate to get $p_3 = 5/6$. However, the second choice $p_1= p_2 = 1/3$, $p_3 = 5/6$ is contradiction to equation system (21). We omit the detail calculation here.

Now we choose $p_1 = p_2 = \frac{1}{2}$ and $p_3 = 1$ to get

$$\begin{cases} b+c+2d=1 \\ \frac{3}{4}b+\frac{3}{4}c+3d=1 \\ \frac{1}{2}b+\frac{1}{2}c+4d=1 \end{cases} \Rightarrow \begin{cases} b+c=\frac{2}{3} \\ d=\frac{1}{6} \end{cases} \Rightarrow \quad a=\frac{1}{6}.$$

Then the equation $6\left(cp_1q_{2b}+dp_1q_{3a}+dp_2q_{3b}\right)=1$ becomes

$$6\left(\frac{1}{2}cq_{2b}+\frac{1}{12}q_{3a}+\frac{1}{12}q_{3b}\right)=1.$$

Because $q_{3a}+q_{3b}=p_3=1$, we get

$$6\left(\frac{1}{2}cq_{2b}+\frac{1}{12}\right)=1 \Rightarrow cq_{2b}=\frac{1}{6}$$

Set $q_{2b}=\frac{1}{t}$ and t as a free parameter. We obtain

$$c=\frac{t}{6} \text{ and } b=\frac{2}{3}-c=\frac{1}{6}(4-t)$$

From the equation

$$4\left(p_1+p_2+6dp_1p_3q_{3a}+6dp_2p_3q_{3b}-6dp_1p_2p_3-24dp_1q_{2b}q_{3b}\right)=1,$$

we get

$$4\left(1+\frac{1}{2}q_{3a}+\frac{1}{2}q_{3b}-\frac{1}{4}-\frac{2}{t}q_{3b}\right)=1.$$

$$4\left(\frac{5}{4}-\frac{2}{t}q_{3b}\right)=1.$$

Therefore $q_{3b}=\frac{t}{2}$ and $q_{3a}=1-\frac{t}{2}$

Then we obtain the general RK4 formula with a parameter t

$$
\begin{cases}
w_{k+1} = w_k + \frac{1}{6}\left[K_1 + (4-t)K_2 + tK_3 + K_4\right]h, \\
K_1 = f(x_k,\ w_k), \\
K_2 = f\left(x_k + \frac{1}{2}h,\ w_k + \frac{1}{2}K_1 h\right), \\
K_3 = f\left(x_k + \frac{1}{2}h,\ w_k + \left(\frac{1}{2} - \frac{1}{t}\right)K_1 h + \frac{1}{t}K_2 h\right), \\
K_4 = f\left(x_k + h,\ w_k + \left(1 - \frac{t}{2}\right)K_2 h + \frac{t}{2}K_3 h\right).
\end{cases}
$$

Moreover, we can check that all constants in this formula satisfy conditions of (21) for any constant t. That proves that is a general Runge-Kutta method in order of 4. Actually, there is no other Runge-Kutta method in order of 4 under the formation (7).

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