Semidiagonals in Quadrilaterals

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Abstract

Relative to a general quadrilateral, semidiagonals are defined. The direct sum of squares of sides and alternating sum of squares of sides are also defined. Several preliminary relationships between the lengths of the sides, the lengths of the semidiagonals, and an angle generated by the intersection of the diagonals are developed. The main results are established which express the direct and alternating sums of squares of sides as functions of the semidiagonals and angle. The angle is then eliminated to express the direct and alternating sums in terms of an initial side and the semidiagonals. These results are simplified for several special cases of quadrilaterals, including the cases of perpendicular diagonals, the parallelogram, the rectangle, and the rhombus. The concluding remarks discuss a failed attempt to derive stronger formulas for the general cases of the direct and alternating sums than are developed in this paper. Finally, an alternative but similar approach to all of the results developed in this paper is presented.

Introduction

Planar figures have fascinated mathematicians as far back as the history of mathematics is recorded. Furthermore, exploring the relationships between the various parts of these figures has captured the attention of many over the years. One of the most basic planar figures is the polygon. The Babylonians studied relationships involving the squares of integer sides of fields for elementary surveying [1, p. 13]. Some of their work is recorded on a clay tablet called *Plimpton 322*, dated back to around 1900 B.C., written in cuneiform (the first known writing), and stored in a museum at Columbia University [1, pp. 12-14]. Relationships between squares of integer sides of polygons were also studied by ancient Egyptians for mostly the same reasons as the Babylonians [1, p. 28]. However, it was the Greeks, in particular the Pythagoreans, who formalized geometry around 550 B.C. and generalized these ideas beyond positive integers [1, pp. 18-19].

Perhaps the most sought after mathematical result in history was Fermat's Last Theorem, which involved the squares of sides of triangles. Posed by Pierre de Fermat around 1637 [1, p. 9], the problem remained unsolved for centuries. It wasn't until 1990 that Ken Ribet published Ribet's Theorem ([1, p. 116],[5]), formerly known as the Epsilon Conjecture. Ribet's Theorem in turn paved the way for Andrew Wiles to eventually complete a proof of the Taniyama-Shimura Conjecture, and consequently Fermat's Last Theorem [1, p. 134], which was published in 1995 [6]. Even now polygons and relationships involving the squares of their sides continue to be of interest.

Basic Definitions

Accordingly, consider now the general quadrilateral Q with vertices V_0, V_1, V_2, and V_3, ordered in a clockwise manner as shown in Figure 1 below. For each integer n, $0 \le n \le 3$, define the side S_n of Q to be the segment between V_n and $V_{n+1(\text{mod}4)}$. Furthermore, Q has diagonals V_0V_2 and V_1V_3 which intersect at a point P. Then for $0 \le n \le 3$, define the *semidiagonal* D_n of Q to be the segment between V_n and P. Finally, define θ to be the angle between the semidiagonals D_0 and D_1. (See Figure 1 below.)

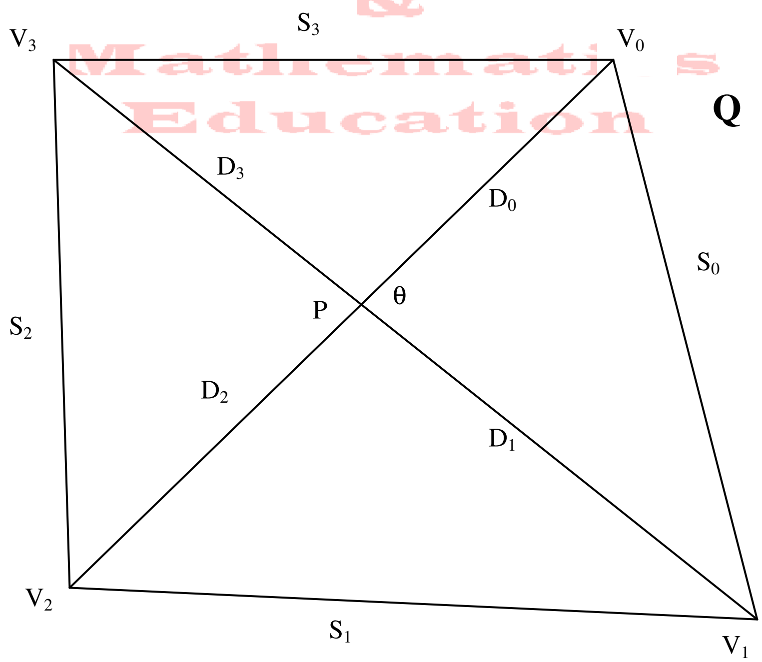

Figure 1

Figure 1 evokes the question of what possible relationships exist between the sides and semidiagonals of Q. Inspired by the work of the Pythagoreans, $\sum_{n=0}^{3} S_n^2$ will be called the *direct sum of squares of sides* of Q, or the *direct sum* of Q for brevity. Similarly, $\sum_{n=0}^{3}(-1)^n S_n^2$ is defined to be the *alternating sum of squares of sides* of Q, or simply the *alternating sum* of Q. We proceed now to establish some initial results upon which the rest of the paper is based.

Preliminary Results

From Figure 1 it is clear that the sides and semidiagonals of Q generate four triangles with common vertex P. Applying the Law of Cosines to each side of Q, we have

$$S_0^2 = D_0^2 + D_1^2 - 2D_0D_1\cos\theta\,, \tag{1}$$

$$S_1^2 = D_1^2 + D_2^2 - 2D_1D_2\cos(180° - \theta) = D_1^2 + D_2^2 + 2D_1D_2\cos\theta\,, \tag{2}$$

$$S_2^2 = D_2^2 + D_3^2 - 2D_2D_3\cos\theta\,, \text{ and} \tag{3}$$

$$S_3^2 = D_3^2 + D_0^2 - 2D_3D_0\cos(180° - \theta) = D_3^2 + D_0^2 + 2D_3D_0\cos\theta\,. \tag{4}$$

Adding (3) to (1) produces $S_0^2 + S_2^2 = (D_0^2 + D_1^2 - 2D_0D_1\cos\theta) + (D_2^2 + D_3^2 - 2D_2D_3\cos\theta)$, and so

$$S_0^2 + S_2^2 = \sum_{n=0}^{3} D_n^2 - 2\cos\theta\left(D_0D_1 + D_2D_3\right). \tag{5}$$

In a similar manner, adding (4) to (2) yields $S_1^2 + S_3^2 = (D_1^2 + D_2^2 + 2D_1D_2\cos\theta) + (D_3^2 + D_0^2 + 2D_3D_0\cos\theta)$, and so

$$S_1^2 + S_3^2 = \sum_{n=0}^{3} D_n^2 + 2\cos\theta\left(D_0D_3 + D_1D_2\right). \tag{6}$$

Adding (6) to (5) produces

$$\begin{aligned} &S_0^2 + S_2^2 + S_1^2 + S_3^2 = \\ &2\sum_{n=0}^{3} D_n^2 - 2\cos\theta\left(D_0D_1 + D_2D_3\right) + 2\cos\theta\left(D_0D_3 + D_1D_2\right) = \\ &2\sum_{n=0}^{3} D_n^2 - 2\cos\theta\left[\left(D_0D_1 + D_2D_3\right) - \left(D_0D_3 + D_1D_2\right)\right] = \\ &2\sum_{n=0}^{3} D_n^2 - 2\cos\theta\left[D_0D_1 + D_2D_3 - D_0D_3 - D_1D_2\right]. \end{aligned}$$

Therefore the direct sum of Q is

$$\sum_{n=0}^{3} S_n^2 = 2\sum_{n=0}^{3} D_n^2 - 2\cos\theta\ (D_0 - D_2)(D_1 - D_3). \tag{7}$$

On the other hand, subtracting (6) from (5) yields

$$\begin{gathered} S_0^2 + S_2^2 - (S_1^2 + S_3^2) = \\ -2\cos\theta(D_0D_1 + D_2D_3) - 2\cos\theta(D_0D_3 + D_1D_2) = \\ -2\cos\theta\ [(D_0D_1 + D_2D_3) + (D_0D_3 + D_1D_2)] = \\ -2\cos\theta\ [D_0D_1 + D_2D_3 + D_0D_3 + D_1D_2]. \end{gathered}$$

Consequently, the alternating sum of Q is

$$\sum_{n=0}^{3} (-1)^n S_n^2 = -2\cos\theta\ (D_0 + D_2)(D_1 + D_3). \tag{8}$$

Main Results

If [[x]] represents the greatest integer less than or equal to x for each real number x, then [[n/2]] = 0 for $0 \le n \le 1$ and [[n/2]] = 1 for $2 \le n \le 3$. Note also that $2 \equiv 0 \pmod 2$ and $3 \equiv 1 \pmod 2$. Substituting these results into (7) and (8) produces alternate formulas

$$\sum_{n=0}^{3} S_n^2 = 2\sum_{n=0}^{3} D_n^2 - 2\cos\theta \cdot \sum_{n\equiv 0 (\mathrm{mod}\, 2)} (-1)^{[[n/2]]} D_n \cdot \sum_{n\equiv 1 (\mathrm{mod}\, 2)} (-1)^{[[n/2]]} D_n \tag{9}$$

and

$$\sum_{n=0}^{3} (-1)^n S_n^2 = -2\cos\theta \cdot \sum_{n\equiv 0 (\mathrm{mod}\, 2)} D_n \cdot \sum_{n\equiv 1 (\mathrm{mod}\, 2)} D_n \tag{10}$$

for the direct and alternating sums of Q, respectively.

In order to eliminate θ from (9) and (10), we solve (1) for the expression $-2\cos\theta$ to obtain

$$-2\cos\theta = \frac{S_0^2 - D_0^2 - D_1^2}{D_0D_1} = \frac{S_0^2 - \sum_{n=0}^{1} D_n^2}{\prod_{n=0}^{1} D_n}. \tag{11}$$

Substituting (11) into (9) and (10) yields the additional formulas

$$\sum_{n=0}^{3} S_n^2 = 2\sum_{n=0}^{3} D_n^2 + \frac{S_0^2 - \sum_{n=0}^{1} D_n^2}{\prod_{n=0}^{1} D_n} \cdot \sum_{n\equiv 0 \pmod 2} (-1)^{[[n/2]]} D_n \cdot \sum_{n\equiv 1 \pmod 2} (-1)^{[[n/2]]} D_n \tag{12}$$

and
$$\sum_{n=0}^{3} (-1)^n S_n^2 = \frac{S_0^2 - \sum_{n=0}^{1} D_n^2}{\prod_{n=0}^{1} D_n} \cdot \sum_{n\equiv 0 \pmod 2} D_n \cdot \sum_{n\equiv 1 \pmod 2} D_n \tag{13}$$

for the direct and alternating sums of Q, respectively.

Several cases in which the quadrilateral Q satisfies various special conditions are worth consideration. We begin with the case in which the diagonals of Q meet at right angles.

Perpendicular Diagonals

If the diagonals of Q are perpendicular then $\theta = 90°$, and so $\cos\theta = 0$. Therefore (9) simplifies to

$$\sum_{n=0}^{3} S_n^2 = 2\sum_{n=0}^{3} D_n^2 \tag{14}$$

for the direct sum of Q. A similar substitution in (10) produces the corresponding formula

$$\sum_{n=0}^{3} (-1)^n S_n^2 = 0 \tag{15}$$

for the alternating sum of Q, which is equivalent to

$$\sum_{n\equiv 0 \pmod 2} S_n^2 = \sum_{n\equiv 1 \pmod 2} S_n^2 . \tag{16}$$

Parallelogram

If Q is a parallelogram, then the diagonals of Q necessarily bisect each other [4, p. 46, Theorem 1.26], so that $D_0 = D_2$ and $D_1 = D_3$. Therefore $\sum_{n\equiv 0 \pmod 2} (-1)^{[[n/2]]} D_n = D_0 - D_2 = 0$ and $\sum_{n\equiv 1 \pmod 2} (-1)^{[[n/2]]} D_n = D_1 - D_3 = 0$. Substituting either of these expressions reduces (9) to

$$\sum_{n=0}^{3} S_n^2 = 2\sum_{n=0}^{3} D_n^2 \tag{17}$$

for the direct sum of Q, which is identical to (14). However, since $D_0 = D_2$ and $D_1 = D_3$, then $\sum_{n\equiv 0 (\mathrm{mod}\, 2)} D_n = 2D_0 = 2D_2$ and $\sum_{n\equiv 1 (\mathrm{mod}\, 2)} D_n = 2D_1 = 2D_3$. Consequently, substitution into (10) for the alternating sum of Q produces the formula

$$\sum_{n=0}^{3} (-1)^n S_n^2 = -8\cos\theta\, D_j\, D_k\,, \tag{18}$$

where $j\in\{0,2\}$ and $k\in\{1,3\}$, or equivalently, $j + k \equiv 1 \pmod 2$.

Rectangle

If the quadrilateral Q is a rectangle, then the diagonals of Q are of equal length ([2, p. 176, Theorem 4.32],[3, p. 141, Theorem 4.33]) and bisect each other [4, p. 46, Theorem 1.26]. Thus $D_i = D_j$ for each $i,j\in\{0,1,2,3\}$, so that $\sum_{n=0}^{3} D_n^2 = 4D_k^2$ for each integer k such that $0 \le k \le 3$. Substitution then reduces the direct sum of Q in (17) to

$$\sum_{n=0}^{3} S_n^2 = 8D_k^2 \tag{19}$$

for each integer k such that $0 \le k \le 3$. However, this result could be obtained with a simple application of the Pythagorean Theorem since adjacent sides of Q are the legs of a right triangle whose hypotenuse is the corresponding diagonal of Q (see Figure 1). Furthermore, since $D_i = D_j$ for each $i,j\in\{0,1,2,3\}$, then $D_i\, D_j = D_k^2$ for each $i,j,k\in\{0,1,2,3\}$. Substitution then reduces the alternating sum of Q in (18) to

$$\sum_{n=0}^{3} (-1)^n S_n^2 = -8\cos\theta\, D_k^2 \tag{20}$$

for each integer k such that $0 \le k \le 3$.

Rhombus

If Q is a rhombus then Q is a parallelogram whose diagonals are perpendicular ([2, p. 178, Theorem 4.35],[3, p. 136, Theorem 4.29]). Therefore formulas (14) through (18) for the direct and alternating sums of Q apply to the rhombus as well. However, the results in (15) and (16) are obvious from the definition of a rhombus. Furthermore, $S_i = S_j$ for each $i,j \in \{0,1,2,3\}$. Finally, since Q is also a parallelogram then $D_0 = D_2$ and $D_1 = D_3$ [4, p. 46, Theorem 1.26]. Thus both (14) and (17) imply that for each integer m such that $0 \le m \le 3$, $4S_m^2 = \sum_{n=0}^{3} S_n^2 = 2\sum_{n=0}^{3} D_n^2 = 2\left(2D_j^2 + 2D_k^2\right) = 4\left(D_j^2 + D_k^2\right)$, where $j \in \{0,2\}$ and $k \in \{1,3\}$. Hence

$$S_m^2 = D_j^2 + D_k^2 \tag{21}$$

for each $m,j,k \in \{0,1,2,3\}$ such that $j + k \equiv 1 \pmod 2$.

Concluding Remarks

An attempt failed to separate the parameters in (12) and (13) so that the sides and semidiagonals of the quadrilateral Q appear only on the left and right sides, respectively, of (12) and (13). More specifically, the initial side S_0 could not be successfully removed from the right side of the equation in either case without introducing other difficulties.

Finally, the derivations above were produced by ordering the vertices, sides, and semidiagonals of Q in a clockwise direction. Clearly the order of these parameters can be assigned just as easily in a counterclockwise manner to produce the same results.

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