

Integer Partitions: An Overview

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Abstract

This paper describes fundamentals of integer partitions and basic tools for their construction and enumeration. In particular, the paper includes some approaches which are not commonly known.

1. Introduction

The mathematics of integer partitions has been studied since the Ancient Greeks. However, most of the existent fundamental results on this subject are the outcome of a number of studies conducted for over 300 years since Leibniz asked Bernoulli if he had investigated the problem to determine the number of partitions of an integer n . In fact, it was Leonhard Euler who made a sustained study of partitions and partition identities, and exploited them to establish a good number of results in Analysis in 1748. However, it was not until 1913 when S. Ramanujan, in collaboration with G. H. Hardy, ingeniously proved several significant results related to integer partitions that the subject of integer partitions picked up [7], for example, contains various historical details). A significant contribution towards development of the mathematics of integer partitions has been due to their applications in combinatorics and algebra ([3] and [4], for example, contain many results). Relatively recently, several competing algorithms to compute both unrestricted and restricted integer partitions have appeared ([17] and [20], for example, provide a good account of algorithms to compute integer partitions along with an extensive list of references on this subject).

A partition of an integer n is its representation as the sum of one or more positive integers where the order in which *summands* appear is immaterial. If the order in which summands appear in an integer partition is taken into consideration, it is called a *composition*. It is immediate to see that two partitions of an integer differ only with respect to the summands they contain.

The central problem concerning integer partitions has been to devise techniques to enumerate distinct number of ways a positive integer n can be expressed as a sum $\sum_{i=1}^k \lambda_i$, where each λ_i belongs to a multiset of positive integers disregarding order. Equivalently, the partitions of a number n can be seen to correspond to the set of solutions $(\lambda_1, \lambda_2, \dots, \lambda_n)$ to the Diophantine equation

Note: A multiset is a collection of objects in which objects can repeat finitely many times. Every individual occurrence of an object in a multiset is called its element. The distinguished elements of a multiset are its objects. The number of times an object appears in a multiset is called its multiplicity. ([16], for example, contains related details).

$$1 \lambda_1 + 2 \lambda_2 + \dots + n \lambda_n = n.$$

For example, one of the partitions of 4 can be seen to correspond to $1 \cdot 4 + 2 \cdot 0 + 3 \cdot 0 + 4 \cdot 0 = 4$, written as $(4, 0, 0, 0)$ which is $(1, 1, 1, 1)$ or $1 + 1 + 1 + 1 = 4$.

In standard representation, a partition λ is any finite or infinite weakly decreasing (or non-increasing) sequence of non-negative integers $(\lambda_1, \lambda_2, \dots, \lambda_k, \dots)$ containing only finitely many non-zero terms, denoted $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k, \dots)$ with $\lambda_1 \geq \lambda_2 \geq \dots$ ([15] and [12]).

Note that some authors choose to use weakly increasing (or non-decreasing) in order to define a partition λ . The non-zero λ_i 's appearing in a partition λ are called its parts (or addends or summands) and the number of parts is called its length, denoted $l(\lambda)$. The sum of the parts of λ is called its weight, denoted $|\lambda|$. If $|\lambda| = n$, then λ is said to be a partition of n , sometimes denoted $\lambda \vdash n$ ([2]).

It is sometimes more useful to represent a partition λ of n in a multiplicity form. For example, a partition $2 + 1 + 1 + 1 + 1$ of 5 in multiplicative form is denoted $(2, 1^4)$ or simply as $(2, 1^4)$.

In general, let $x_1 > x_2 > \dots > x_d$ be the distinct parts in a partition of n and m_1, m_2, \dots, m_d their corresponding multiplicities where each m_i is a positive integer, then $n = m_1 x_1 + m_2 x_2 + \dots + m_d x_d$ is represented $(x_1^{m_1}, x_2^{m_2}, \dots, x_d^{m_d})$. This is similar to the frequency representation of partition of n ([2]). There are other notations occasionally found useful ([12]).

The enumerator function representing the number of all unrestricted partitions of weight n , denoted P_n or $P(n)$, is called the partition function of n . For example,

$$P(4) = 5 \quad \text{viz.,} \\ \{4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1\}.$$

Conventionally, $P(0) = 1$, $P(n) = 0$ for a negative n . The empty partition \emptyset is viewed as the unique partition of zero and hence $P(0) = 1$.

In the literature dealing with integer partitions, some typical questions concerning $P(n)$ that have continually been receiving attention include the following:

- (i) How fast does $P(n)$ grow?

- (ii) Are there efficient ways to compute $P(n)$?
- (iii) What is its parity?
- (iv) What are its distinctive arithmetic properties?
- (v) Is $P(n)$ prime infinitely often?

All these problems are known to be quite difficult. For example, let us consider the problem (i) mentioned above. The first few empirical results viz., $P(1) = 1, P(2) = 2, P(3) = 3, P(4) = 5, P(5) = 7, P(6) = 11$ may suggest that $P(n)$ runs through prime numbers, but it is false because $P(7) = 15, P(8) = 22, P(9) = 30, P(10) = 42, \dots, P(1000) = 2.4 \times 10^{24}$, etc.

As of April 2011, the largest known prime of the kind is $P(30248445)$ with 6119 decimal digits, found by Bernardo Boncompagni [6].

A nice formulation of this problem is provided in ([10], P.32):

$$\sum a_{j_1} \dots j_n = n,$$

$$j_1 + j_2 + \dots + j_n = n,$$

$$j_1 \geq j_2 \geq \dots \geq j_n \geq 0,$$

where a is an n -tuply subscripted variable gives a representation of a partition of n . For example, if $n = 5$, the aforesaid notation stands for

$$a_{11111} + a_{21110} + a_{32100} + a_{41100} + a_{50000} + a_{33000} + a_{41000} + a_{50000}.$$

Notwithstanding, in 1918, it was ingenious S. Ramanujan, in collaboration with G. H. Hardy, besides providing a number of remarkable results about $P(n)$, succeeded for the first time in answering the question (i) above by discovering an asymptotic characterization of $P(n)$ viz. ,

$$P(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}} \text{ as } n \rightarrow \infty, \text{ which shows that the growth of } P(n) \text{ is subexponential.}$$

J. V. Usperky obtained the same result independently in 1920.

The striking feature of the aforesaid formulation lies in the fact that it provides results reasonably quite close to the exact solution; for example,

$$P(1000) = 2.4402 \times 10^{24} \text{ as against } 2.4 \times 10^{24}.$$

In 1937, Hans Rademacher [13] was able to improve on the aforesaid formulation by providing a convergent series expression for $P(n)$.

Question (ii) mentioned above was first attacked by the great Swiss mathematician Leonhard Euler [9] who discovered the following recurrence relation for $P(n)$:

$$P(k) = P(k-1) + P(k-2) - P(k-5) - P(k-7) + P(k-12) + P(k-15) - P(k-22) - P(k-26) + \dots$$

where $P(k) = 1$ if $k = 0$, $P(k) = 0$ if k negative, and the sum is taken over all generalized pentagonal numbers of the form $\frac{1}{2}n(3n-1)$, for $n = 1, -1, 2, -2, 3, -3, \dots$ in succession.

As to the performance of the aforesaid algorithm, Andrews [2] and Andrews and Kimino [3] note:

No one has ever found a more efficient algorithm for computing $P(n)$. It computes a full table of values of $P(n)$ for $n \leq N$ in time $O(N^{3/2})$.

Percy Macmahon, by exploiting the aforesaid recurrence relation, constructed a table of $P(n)$ for $n = 1, 2, \dots, 200$. Gupta (1939) published a more extensive table of $P(n)$ for $1 \leq n \leq 600$ ([7]).

Euler also used the recurrence relation

$$P(n, m) = P(n, m-1) + P(n-m, m), \quad n \geq 1$$

$$P(0, m) = 1$$

to determine $P(n, m)$ for increasing n for each m , which helped deriving $P(n)$ viz., $P(n, m) = P(n)$ for $m \geq n$. Also, a table of $P(n, m)$ for $n \leq 69$ and $m \leq 1$ was constructed.

Note that the number of partitions of n into at most m parts is usually denoted $P(n, m)$.

It is noted [7] that Gupta (1939) also used the recurrence relation $r(n, m) = r(n, m+1) + r(n-1, m)$

to determine $r(n, m)$, defined as the number of partitions of n into parts, the smallest being at least m , which helped deriving $P(n)$ viz., $r(n, 1) = P(n)$.

As for now, questions (iii) and (v) are still largely open, while the question (iv) has been answered in the affirmative. A full discussion of the questions (iii) and (iv) can be found in [2] and [3].

2.0 Two basic tools for dealing with Partitions of Integers

2.1 Ferrers diagram: The scheme for representing a partition by a diagram is attributed to Ferrers and Sylvester, often called Ferrers diagram (or Young diagram). The Ferrers diagram corresponding to a partition λ , denoted by F_λ , is a way of visualizing λ with a graphical representation comprising of a schematic arrangement of dots (or blocks). Basically, the construction of the Ferrers diagram for a given portion $\lambda = \{\lambda_1, \lambda_2, \dots, \lambda_k\}$ requires placing a row of λ_{i+1} left justified blocks on top of λ_i blocks, for each $i = 1, 2, \dots, k-1$. Each row represents one addend in the partition. The number of blocks in a row represents the value of that total addend. For example, the Ferrers diagram F_λ for the partition $\lambda = (5, 3, 1, 1)$ is depicted below:

Fig. 1 (Stair-shaped stacking of squares or dots)

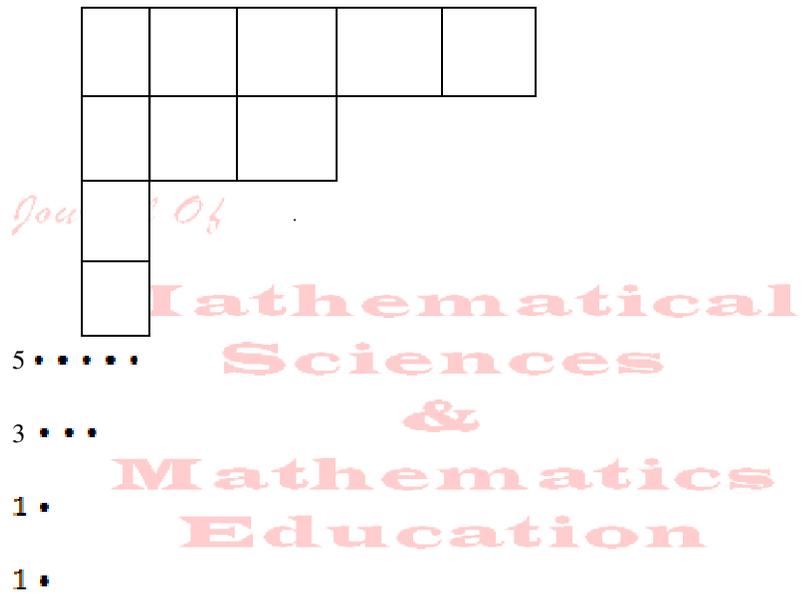
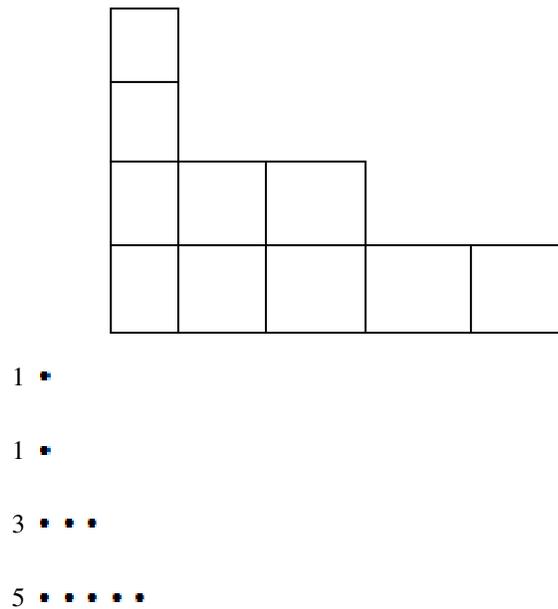


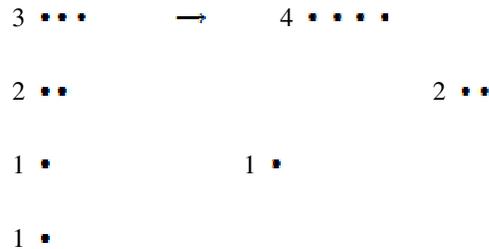
Fig. 2, Alternatively, the above representations can be depicted upside-down as follows:



Conventionally, any one of them can be selected and in each case there is a bijection between Ferrers diagram with k blocks and partitions of n ([12], for example, contains details). It is noteworthy that many intriguing results about restricted partitions have been established with the aid of Ferrers diagram ([2], for example, contains contributions made by Sylvester (1984-86), Macmohan (1916) and others, including his own, in this regard). Moreover, the possibilities for new results to be discovered seem promising.

2.2 Some typical properties of Ferrers diagrams are as follows: [proofs can be found in various texts dealing with combinatorics]:

(i) If a Ferrers diagram of a partition is reflected about the 45° downward slanting line through the upper left dot, we obtain the same partition as when we count the dots by the columns rather by rows. The resulting diagram is called the *conjugate* of the original diagram. For example,



That is, the conjugate of $(3, 2, 1, 1)$ is $(4, 2, 1)$. The following result is immediate by reasoning on Ferrers diagram and conjugate partitions:

The number of partitions of n with k parts equals the number of partitions of n whose largest part is k .

In general, the conjugate of a partition λ is denoted by λ' and the diagram of λ' is obtained by taking the transpose of the diagram of λ .

A partition is called *self-conjugate* if it equals its conjugate; or, equivalently, if the Ferrers diagram is symmetric about its diagonal. An interesting result in this regard is as follows:

There is a bijection between self-conjugate partitions and partitions all whose parts are odd and distinct. In other words, the number of self-conjugate partitions of n equals the number of partitions of n all whose parts are odd and distinct.

(ii) The number of partitions of n into parts, the largest of which is r , is equal to the number of partitions of n into exactly r parts. To illustrate this fact, let us consider the original diagram of the partition $(3, 2, 1, 1) = 7$ in which the largest part $(r) = 3$ and reading it columnwise, we obtain the partition viz.,

4 2 1



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with exactly $r = 3$ parts of the same number $n = 7$.

(iii) It is expedient to observe the occurrence of the largest square embedded in the Ferrers diagram of a partition λ . This is called the *Durfee square* of the partition λ . The number of boxes on the main diagonal of the Ferrers diagram F_λ is called the *Durfee size* of the partition and defined by $d(\lambda)$. Formally, $d(\lambda)$ is the largest value i such that $\lambda_i \geq i$. Thus, the Durfee square of λ is the subpartition built from the $d(\lambda) \times d(\lambda)$ boxes. For example, in the Ferrers diagram given above in (ii), $d(\lambda) = 2$ and Durfee square of λ is of the size 2×2 .

2.3 Generating Functions

As noted in ([10], p.86), it was Pierre Laplace who introduced generating functions in his classic work *Theorie Analytique des probabilités* (1812), and part of the credit for discovering generating functions goes to Bernoulli (1728) and J. Stirling (1730). However, Leonhard Euler was the first ([10], p.494) to apply generating functions particularly to count the partitions by size. Euler proved the following theorem:

For $q > 1$, $\sum_{n \geq 0} P(n) q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}$ (proof can be found in [2]).

As noted in [7], Euler, while providing solution to a problem proposed by Ph. Maude, asserted that the number of partitions of n into m distinct parts is the coefficient of $x^m q^n$ in the power series expansion of the infinite product $\prod_{j=1}^{\infty} (1 + xq^j)$.

The motivation for exploiting generating functions to solve partition problems lies in the fact that they can be manipulated more easily than their combinatorial counterparts. The central idea for constructing a generating function for a given sequence or numeric function $(a_0, a_1, a_2, \dots, a_k, \dots)$ consists in

formulating a *power series* (or an *analytic function*) in terms of a parameter z viz.,

$$A(z) = \sum_{k=0}^{\infty} a_k z^k = a_0 + a_1 z + a_2 z^2 + \dots,$$

where the coefficient a_k of z^k represents the number of ways that the event k can occur. That is, the coefficients encode information about the sequence (a_k) indexed by the natural numbers, put metaphorically by Wilf [19] in the following words:

A generating function is a clothesline on which we hang up a sequence of numbers for display.

The generating function for the sequence (a_k) described above is sometimes called the *ordinary generating function* of the sequence (a_k) . If it is used for solving counting problems, it is called *formal power series*. A typical example is to construct a generating function for the number of subsets of an n -element set viz., $S(z) = 1 + 2z + 2^2 z^2 + \dots + 2^n z^n + \dots$ which may be viewed as a generating function $(1, 2^1, 2^2, \dots, 2^n, \dots)$ where the coefficient 2^n of z^n is the number of subsets of an n -element set. Essentially, the terms of the sequence (2^n) are coded as the coefficients of powers of a variable z in power series of the form described above.

Note that the infinite series viz., $1 + 2z + 2^2 z^2 + \dots$ can be written as $\frac{1}{1-2z}$, assuming $|z| < 1$ for which the series converges. However, in order to solve the problems related to counting of partitions deploying a formal power series expansion, we need not be concerned with the convergence issue of the series. Essentially, it is the coefficient of z^k in a particular expansion that serves the purpose.

In turn, it is suggested that the problem of counting partitions of a number by size can be tackled by constructing an appropriate formal geometric series, guided by the fact that one needs to know how many ones, twos, threes and so on are there in the partitions. Since in each partition, one or two or three and so on can occur 0, 1, 2, ... times respectively, the contributing factors to the generating function will be $(1 + x + x^2 + \dots)$ corresponding to number of occurrences of one, $(1 + x^2 + x^4 + \dots)$ for two, $(1 + x^3 + x^6 + x^9 + \dots)$ for three, and so on. Accordingly, the geometric series $\frac{1}{1-x}, \frac{1}{1-x^2}, \dots, \frac{1}{1-x^n}, \dots$ with $|x| < 1$ constitutes the generating function for the number of partitions by size viz., $\sum_{n \geq 0} P(n) x^n$ where $P(n)$ is the number of partitions of n , which is given by $\prod_{n \geq 1} \frac{1}{1-x^n} = \frac{1}{(1-x)(1-x^2)(1-x^3)} \dots$. That is, the number of partitions of n equals the coefficient $P(n)$ of x^n in the formal power series expansion. Moreover, the number of partitions of n into odd integers (i.e., the number of ways to write n as the sum of odd positive integers) where the order is immaterial and repetitions are allowed) is equal to the coefficient $(P_o(n))$, say of x^n in the formal power series expansion of $\frac{1}{(1-x)(1-x^3)(1-x^5)} \dots$. Also, the number of partitions of n into distinct parts (i.e., the number of ways to write n as the sum of positive integers), where the order is immaterial but no repetitions

allowed is equal to the coefficient ($P_d(n)$, say) of x^n in the formal power series expansion

$$(1+x)(1+x^2)(1+x^3)\dots$$

It is known [Euler] that the generating functions for $P_o(n)$ and $P_d(n)$ are the same i.e.,

$$P_o(n) = P_d(n), \text{ since}$$

$$\frac{(1+x)(1+x^2)(1+x^3)\dots}{1-x} = \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \dots$$

Alternatively, the proof can be easily obtained as follows:

The generating function for the number $D(n)$ of partitions of n into distinct parts can be given by

$$\sum_{n \geq 0} D(n) q^n = \prod_{n=1}^{\infty} (1+q^n).$$

The generating function for the number $O(n)$ of partitions of n into odd parts can be given by

$$\sum_{n \geq 0} O(n) q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^{2n-1}}.$$

Since $(1-q^{2n}) = (1-q^n)(1+q^n)$, it follows that

$$\sum_{n \geq 0} D(n) q^n \prod_{n=1}^{\infty} (1-q^n) = \prod_{n=1}^{\infty} (1-q^{2n})$$

or,

$$\sum_{n \geq 0} D(n) q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^{2n-1}} = \sum_{n \geq 0} O(n) q^n.$$

For instance, if $n = 8$, the two classes of partitions are given respectively by (4 3 1), (5 2 1), (5 3), (6 2), (7 1), 8 and (1⁸), (3 1⁵), (3² 1³), (5 1³), (5 3), (7 1).

In fact, many similar partition identities taking some sort of generalizations of the stipulation (*distinctness*, for example) of Euler's theorem (for example, Rogers-Ramanujan identities) have been discovered ([2] contains proofs and various other details). The most striking result in this regard is that of Bousquet-Mélou and Erickson [5] who interpreted *distinctness* condition of Euler's theorem as a *multiplicative* property rather than an *additive* property and developed the theory of *Lecture Hall Partitions*.

Remark: It may be observed that the geometric series mentioned above builds up Ferrers diagrams. For example,

$$\frac{1}{1-x} = \frac{1}{1-x} = 1 + \begin{array}{|c|} \hline \square \\ \hline \end{array} + \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} + \dots$$

$$\text{or, } \frac{1}{1-x^2} = \frac{1}{1-x^2} = 1 + \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array} + \dots$$

builds up Ferrers diagram by rows; and

$$\frac{1}{1-x} = \frac{1}{1-x} = 1 + \begin{array}{|c|} \hline \square \\ \hline \end{array} + \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} + \dots$$

$$\text{or, } \frac{1}{1-x^2} = \frac{1}{1-\square\square} = 1 + \square + \square\square + \square\square\square + \dots$$

builds up Ferrers diagrams by columns.

Moreover, the multiplication of two or more series of the above form corresponds to the juxtaposition of rows or columns in building up Ferrers diagrams.

Before closing our discussion on this topic, we wish to mention that a very convenient method to enumerate the number of partitions of m using a generating function is provided in ([10], p.92) as follows:

$$G(x) = \sum_{m \geq 0} \left(\sum_{k_1, k_2, \dots, k_m \geq 0} \frac{x_1^{k_1}}{k_1!} \frac{x_2^{k_2}}{2^{k_2} k_2!} \dots \frac{x_m^{k_m}}{m^{k_m} k_m!} \right)^m x^m$$

where $(k_1 + 2k_2 + \dots + mk_m = m)$.

The parenthesized quantity is a_m . The number of terms for a particular value of m is $P(m)$, the number of partitions of m . For example, one partition of 12 is

$$12 = 1 + 2 + 2 + 2 + 5,$$

which corresponds to a solution of the equation $k_1 + 2k_2 + \dots + 12k_{12} = 12$, where k_j is the number of j 's in the partition. In the example above,

$k_1 = 1, k_2 = 3, k_5 = 1$ and the other k 's are zero, giving the expansion

$$\frac{x_1}{1!} \frac{x_2^3}{2^3 3!} \frac{x_5}{5!} = \frac{1}{240} S_1 S_2^3 S_5$$

as a part of the expansion for a_{12} . It may be noted at this end that there are many other applications of the generating functions which have not been addressed in this paper; for example, solving recurrence relations, proving a number of *combinatorial* identities, etc., ([2], [3], [11], [12] and [19], for example, contain many such partition theoretic applications of generating functions).

It is also worthy of notice that multiple-variable generating functions are also exploited in enumerating certain classes of partitions, and in proving deeper partition identities ([1], [2] and [14], for example, contain a variety of results in this regard).

Appendix

The following are some frequently used generating functions for simple numeric functions:

$$A(x) = \frac{1}{1-x} \text{ for the numeric function with its general term } a_r = 1$$

$$A(x) = \frac{x}{1-x^2} \text{ for the numeric function with its general term } a_r = r.$$

$$A(x) = \frac{x^2}{(1-x)^3} \text{ for the numeric function with its general term } a_r = r(r+1).$$

$$A(x) = \frac{1}{1-\alpha x} \text{ for the numeric function with its general term } a_r = \alpha^r,$$

where α is a constant.

$$(i) A(x) = (1+x)^n \text{ for the numeric function with its general term}$$

$$a_r = \binom{n}{r}.$$

$A(x) = \frac{x(2+x)}{(1-x)^3}$ for the numeric function with its general term $a_r = r^2$.

$A(x) = e^x$ for the numeric function with its general term $a_r = \frac{1}{r!}$.

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