# Semidisjoint Permutations

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**Abstract**

Fixed points and transient points are defined in the group Sym(S) of permutations on a nonempty set S. Disjoint and semidisjoint permutations are then defined in terms of transient points. It is shown that the collection of disjoint pairs of permutations in Sym(S) is (properly) contained in the collection of semidisjoint pairs of permutations in Sym(S). Two main commutativity results for semidisjoint permutations are established. A counterexample is provided to verify that these two results cannot be combined to produce a more general commutativity result for semidisjoint permutations which has already been established for disjoint permutations.



## Introduction

For a nonempty set S, it is commonly known that Sym(S) endowed with the operation of composition of functions is a group [2, p. 38, Theorem 6.1], called the group of permutations on S. It is also well known that Sym(S) is nonabelian whenever $|S| \geq 3$ ([1, p. 94, Theorem 2.20],[2, p. 40, Theorem 6.3]). Therefore any result on commutativity in nontrivial permutation groups is significant. A standard commutativity result regarding disjoint permutations appears in most relevant texts. However, the degree of generality of this result varies substantially in the literature. In its weakest form, it is stated that disjoint pairs of cycles on a finite nonempty set commute ([1, p. 95],[2, p. 41],[3, p. 82, Theorem 6.2],[4, p. 131, Lemma 3.2.1]). This statement is sometimes generalized by extending the class of permutations from cycles to arbitrary permutations, while leaving the underlying nonempty set finite. The resulting claim is that disjoint pairs of arbitrary permutations on a finite nonempty set commute [5, p. 47]. Still other sources generalize the weak form by extending the set on which the permutations are defined from a finite nonempty set to an arbitrary nonempty set, while leaving the relevant permutations restricted to cycles. Thus the resulting statement is that disjoint pairs of cycles on an arbitrary nonempty set commute [6, p. 79, no. 10].

These two generalizations have been combined to produce the more comprehensive result that disjoint permutations in general on an arbitrary nonempty set commute [7, Theorem 9]. This paper will define the notion of semidisjoint permutations and show that it generalizes the concept of disjoint permutations. Two commutativity results for semidisjoint permutations will be developed which correspond to the special cases described above for disjoint permutations ([5, p. 47],[6, p. 79, no. 10]). It will then be verified that these results cannot be extended to include general semidisjoint permutations on an arbitrary nonempty set as was done for disjoint permutations [7, Theorem 9]. Throughout this paper it is assumed that S is a nonempty set.

## Basic Definitions

We begin with some fundamental definitions and notations which are pertinent to all of the following results. The initial concepts of permutations, Sym(S), $S_n$, cycles, and the identity map on S are standard [7, Definition 1]. However, they are included here for completeness.

**Definition 1**: If S is a nonempty set, then a permutation (or symmetry) $\alpha$ on S is a 1-1, onto function $\alpha$:S$\rightarrow$S. The set of all permutations on S is denoted by Sym(S). If S is a finite set of order n, then Sym(S) will be written $S_n$, and is called the set of permutations on n elements. In this case S can be represented as S = $\{k\}_{k=1}^{n}$. If $\alpha \in$ Sym(S) and n is a positive integer, then $\alpha$ is a cycle of length (or order) n if and only if there is a finite subset $\{a_i\}_{i=1}^{n}$ of S such that $\alpha(a_i)$ = $a_{i+1}$ for $1 \le i \le n-1$, $\alpha(a_n) = a_1$, and $\alpha(x) = x$ for each $x \in S - \{a_i\}_{i=1}^{n}$. In this case we write $\alpha = (a_1, a_2, \cdots, a_n)$. The identity map on S is denoted by $1_S$.

The definitions of fixed and transient points have been previously established [7, Definition 2]. Nevertheless, they are provided here since the primary definition and main results of the paper directly depend on them.

**Definition 2**: Suppose S is a nonempty set, $p \in S$, and $\alpha \in$ Sym(S). Then p is a fixed point of $\alpha$ if and only if $\alpha(p) = p$; p is a transient point of $\alpha$ if and only if $\alpha(p) \neq p$. The set of fixed points of $\alpha$ is $F_\alpha = \{x \in S \mid \alpha(x) = x\}$; the set of transient points of $\alpha$ is $T_\alpha = \{x \in S \mid \alpha(x) \neq x\}$.

In 2011 disjoint permutations, disjoint cycles, and disjoint collections of permutations were defined in terms of transient points [7, Definition 7]. However, one major goal of this paper is to verify that the collection of disjoint pairs of permutations on S is contained (properly in all but the most trivial case) in the collection of semidisjoint pairs of permutations on S. Hence these definitions are repeated here due to their critical nature.

**Definition 3**: Suppose $\alpha,\beta \in$ Sym(S). Then $\alpha$ and $\beta$ are disjoint if and only if $T_\alpha \cap T_\beta = \varnothing$. Consequently, if $\alpha = (a_1, \cdots, a_k)$ and $\beta = (b_1, \cdots, b_m)$ are cycles in Sym(S), then $\alpha$ and $\beta$ are disjoint if and only if $a_i \neq b_j$ for each i and j such that $1 \le i \le k$ and $1 \le j \le m$. A collection C of permutations in Sym(S) is disjoint if and only if $\alpha$ and $\beta$ are disjoint for each $\alpha,\beta \in C$ such that $\alpha \neq \beta$.

A quick observation provides some perspective on the phrase “disjoint permutations”. If $\alpha,\beta \in$ Sym(S), then by Definition 2 and Definition 3, $\alpha$ and $\beta$

are disjoint as permutations if and only if their respective collections $T_\alpha$ and $T_\beta$ of transient points are disjoint as sets ([1, p. 95],[6, p. 79]).

We now introduce the main notion of semidisjoint permutations in Sym(S). It will be shown that this concept generalizes that of disjoint permutations. Furthermore, certain previously established results [7] related to disjoint permutations will be extended to include semidisjoint permutations as well.

**Definition 4**: Suppose $\alpha,\beta \in$ Sym(S). Then $\alpha$ and $\beta$ are semidisjoint if and only if $\alpha(x) = \beta(x)$ for each $x \in T_\alpha \cap T_\beta$. A collection C of permutations in Sym(S) is semidisjoint if and only if $\alpha$ and $\beta$ are semidisjoint for each $\alpha,\beta \in C$.



## Preliminary Results

Some important distinctions between disjoint and semidisjoint permutations should be noted. Semidisjoint permutations $\alpha$ and $\beta$ do not require that $T_\alpha \cap T_\beta = \varnothing$ as do disjoint permutations. Instead there is the weaker requirement only that $\alpha(x) = \beta(x)$ for each $x \in T_\alpha \cap T_\beta$. A second and somewhat more subtle difference is that the definition of a semidisjoint collection makes no mention of the condition that $\alpha \neq \beta$ as appears in the definition of a disjoint collection. The reason for this last distinction is revealed in the following corollary, and will be useful for later results.

**Corollary 5**: If $\alpha \in$ Sym(S), then $\alpha$ is semidisjoint with itself.

Proof: Clearly $\alpha(x) = \alpha(x)$ for each $x \in T_\alpha \cap T_\alpha$. Hence $\alpha$ and $\alpha$ are semidisjoint by Definition 4.

It is noteworthy that the result in Corollary 5 is not necessarily true for disjoint permutations. In fact, a permutation in Sym(S) is not disjoint with itself except in the trivial case of the identity map $1_S$ [7, Corollary 8].

We are now prepared to show that disjoint permutations are a special case of semidisjoint permutations. The following three results establish this fact.

**Corollary 6**: Suppose $\alpha,\beta \in$ Sym(S). If $\alpha$ and $\beta$ are disjoint, then $\alpha$ and $\beta$ are semidisjoint.

Proof: If $\alpha$ and $\beta$ are disjoint, then $T_\alpha \cap T_\beta = \varnothing$ by Definition 3. Thus it is vacuously true that $\alpha(x) = \beta(x)$ for each $x \in T_\alpha \cap T_\beta$. Hence $\alpha$ and $\beta$ are semidisjoint by Definition 4.

Alternatively, if $\alpha$ and $\beta$ are not semidisjoint, then by Definition 4 there exists some $x \in T_\alpha \cap T_\beta$ such that $\alpha(x) \neq \beta(x)$. Therefore $T_\alpha \cap T_\beta \neq \varnothing$, and so $\alpha$

and $\beta$ are not disjoint by Definition 3. The result follows immediately from the contrapositive.

The result in Corollary 6 for pairs of permutations in Sym(S) can easily be extended to collections of permutations in Sym(S).

**Corollary 7**: If C is a disjoint collection of permutations in Sym(S), then C is a semidisjoint collection of permutations in Sym(S).

Proof: Suppose C is a disjoint collection in Sym(S) and $\alpha,\beta \in C$. Then either $\alpha = \beta$ or $\alpha$ and $\beta$ are disjoint by Definition 3. If $\alpha = \beta$, then $\alpha$ and $\beta$ are semidisjoint by Corollary 5. Otherwise $\alpha$ and $\beta$ are disjoint, and thus are semidisjoint by Corollary 6. Therefore C is a semidisjoint collection in Sym(S) by Definition 4.



Stated differently, Corollary 6 shows that the collection of all disjoint pairs of permutations on S is contained in the collection of all semidisjoint pairs of permutations on S. However, when S is a nontrivial nonempty set, the actual relationship between these two collections is that of *proper* set containment. By establishing this result, it is determined that the converse of Corollary 6 is false. In other words, there exist semidisjoint pairs of permutations which are not disjoint.

**Theorem 8**: Suppose H and K are the collections of all disjoint and semidisjoint pairs of permutations on S, respectively. Then $H \subseteq K$. Furthermore, $H \subset K$ if and only if $|S| > 1$.

Proof: If $\{\alpha,\beta\} \in H$ then $\alpha$ and $\beta$ are disjoint. Therefore $\alpha$ and $\beta$ are semidisjoint by Corollary 6, and so $\{\alpha,\beta\} \in K$. Hence $H \subseteq K$.

Since S is a nonempty set, then $|S| \geq 1$. If $|S| = 1$, then $\text{Sym}(S) = S_1 = \{1_S\}$. Furthermore, $1_S$ is disjoint with itself [7, Corollary 8]. Consequently $1_S$ is semidisjoint with itself by Corollary 6. (Alternatively, $1_S$ is semidisjoint with itself by Corollary 5.) Hence $H = \{\{1_S,1_S\}\} = K$.

On the other hand, if $|S| > 1$ then there exist $p,q \in S$ such that $p \neq q$. Define the cycle $\alpha = (p,q)$, so that $T_\alpha = \{p,q\}$ by Definition 1 and Definition 2. Then $\alpha$ is semidisjoint with itself by Corollary 5, and so $\{\alpha,\alpha\} \in K$. However, it is clear that $\alpha = (p,q) \neq 1_S$, and so $\alpha$ is not disjoint with itself [7, Corollary 8]. (Alternatively, since $T_\alpha \cap T_\alpha = \{p,q\} \neq \varnothing$, then $\alpha$ is not disjoint with itself by Definition 3.) Therefore $\{\alpha,\alpha\} \notin H$, and so $H \subset K$.

Corollary 6 confirmed that if $\alpha,\beta \in \text{Sym}(S)$ and $\alpha$ and $\beta$ are disjoint, then $\alpha$ and $\beta$ are semidisjoint. In contrast, Theorem 8 verified that permutations in Sym(S) may be semidisjoint but not disjoint. For example, a permutation $\alpha$ is considered to be semidisjoint with itself by Corollary 5. However, $\alpha$ is not disjoint with itself if $T_\alpha \neq \emptyset$, or equivalently, if $\alpha \neq 1_S$ [7, Corollary 8]. That is, if $\alpha = \beta$ then $\alpha$ and $\beta$ are semidisjoint but not necessarily disjoint.

Although it is not true for permutations in general, this is the only way in which cycles in particular can be semidisjoint but not disjoint. That is, it is impossible for two distinct cycles to be semidisjoint but not disjoint. Consequently, we now show that semidisjoint cycles in Sym(S) are either disjoint or identical.

**Lemma 9**: If $\alpha$ and $\beta$ are semidisjoint cycles in Sym(S), then either $\alpha$ and $\beta$ are disjoint or $\alpha = \beta$.

Proof: Suppose $\alpha$ and $\beta$ are semidisjoint cycles in Sym(S) which are not disjoint. If $\alpha$ is a cycle of length $r = 1$ then $\alpha = 1_S$, so that $T_\alpha = \emptyset$. Then $T_\alpha \cap T_\beta = \emptyset$, so that $\alpha$ and $\beta$ are disjoint by Definition 3. This is a contradiction, and so $r > 1$.

Since $\alpha$ and $\beta$ are not disjoint, then $T_\alpha \cap T_\beta \neq \emptyset$ by Definition 3, so there exists some $a_1 \in T_\alpha \cap T_\beta$. Furthermore, since $r > 1$ then there exists $\{a_i\}_{i=2}^{r} \subseteq S$ such that $\alpha = (a_1, a_2, a_3, \cdots, a_r)$, where $a_i \neq a_j$ whenever $1 \leq i < j \leq r$, $\alpha(a_i) = a_{i+1}$ for $1 \leq i \leq r-1$, and $\alpha(a_r) = a_1$ by Definition 1.

Since $\alpha$ and $\beta$ are semidisjoint and $a_1 \in T_\alpha \cap T_\beta$, then $\beta(a_1) = \alpha(a_1) = a_2$ by Definition 4, and so $a_2 \in T_\alpha \cap T_\beta$ ([7, Corollary 5(a)],[7, Corollary 6(b)]). Similarly, since $\alpha$ and $\beta$ are semidisjoint and $a_2 \in T_\alpha \cap T_\beta$, then $\beta(a_2) = \alpha(a_2) = a_3$ by Definition 4, and so $a_3 \in T_\alpha \cap T_\beta$ ([7, Corollary 5(a)],[7, Corollary 6(b)]).

Continuing in this manner, we have $a_{i+1} = \alpha(a_i) = \beta(a_i)$ for $1 \leq i \leq r-1$, and $a_i \in T_\alpha \cap T_\beta$ for $1 \leq i \leq r$. Since $\alpha$ and $\beta$ are semidisjoint and $a_r \in T_\alpha \cap T_\beta$, then $\beta(a_r) = \alpha(a_r) = a_1$ according to Definition 4. As a result, $\beta$ is also a cycle of length r. Furthermore, $\alpha = (a_1, a_2, a_3, \cdots, a_r) = (a_1, \alpha(a_1), \alpha(a_2), \cdots, \alpha(a_{r-1})) = (a_1, \beta(a_1), \beta(a_2), \cdots, \beta(a_{r-1})) = \beta$.

It is noteworthy that the result in Lemma 9 cannot be extended to include arbitrary permutations in Sym(S). In other words, there exist semidisjoint permutations $\alpha, \beta \in$ Sym(S) which are not disjoint and for which $\alpha \neq \beta$. In fact, the result in Lemma 9 cannot be guaranteed even in the case involving one cycle and one arbitrary permutation on a finite set. Consider the example in the concluding remarks of [7] in which $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$. We have $\alpha, \beta \in S_4$ and $\alpha = (1,2)$ is a cycle, but $\beta$ is not a cycle. Therefore $T_\alpha = \{1,2\}$ and $T_\beta = \{1,2,3,4\}$, so that $T_\alpha \cap T_\beta = \{1,2\}$. Since $\alpha(1) = 2 = \beta(1)$ and

$\alpha(2) = 1 = \beta(2)$, then $\alpha$ and $\beta$ are semidisjoint by Definition 4. However, $T_\alpha \cap T_\beta \neq \varnothing$, so that $\alpha$ and $\beta$ are not disjoint by Definition 3. Furthermore, it is clear that $\alpha \neq \beta$ since $\alpha(3) = 3$ but $\beta(3) = 4$. A much more sophisticated example of this and other properties of semidisjoint permutations is provided in the concluding remarks.

Lemma 9 completes the prerequisites necessary to establish the first main result on commutativity for semidisjoint permutations. We now proceed to develop the final material critical for the second main commutativity result. More specifically, the following lemma establishes the fact that when two semidisjoint permutations can be factored as finite products of distinct disjoint permutations, then each pair of their combined factors inherits the property of being semidisjoint.



**Lemma 10**: Suppose that $\alpha,\beta \in \mathrm{Sym}(S)$, $\alpha$ and $\beta$ are semidisjoint, $\alpha = \alpha_1 \cdots \alpha_k$, $\beta = \beta_1 \cdots \beta_m$, and each of $\{\alpha_i\}_{i=1}^{k}$ and $\{\beta_i\}_{i=1}^{m}$ is a disjoint collection of distinct permutations in Sym(S). Then $\alpha_r$ and $\beta_j$ are semidisjoint for each r and j such that $1 \le r \le k$, $1 \le j \le m$. Furthermore, $\{\alpha_i\}_{i=1}^{k} \cup \{\beta_i\}_{i=1}^{m}$ is a semidisjoint collection of permutations in Sym(S).

Proof: Suppose that $1 \le r \le k$. Since $\{\alpha_i\}_{i=1}^{k}$ is a disjoint collection of distinct permutations in Sym(S), then $\alpha_r$ and $\alpha_s$ are disjoint whenever $r \neq s$. Thus if $x \in T_{\alpha_r}$, then $x \notin T_{\alpha_s}$ whenever $s \neq r$ by Definition 3. Therefore $x \in F_{\alpha_s}$ whenever $s \neq r$ [7, Corollary 3], and so $\alpha_s(x) = x$ for each $s \neq r$ by Definition 2. Furthermore, since $x \in T_{\alpha_r}$, then $\alpha_r(x) \in T_{\alpha_r}$ ([7, Corollary 5(a)],[7, Corollary 6(b)]). By an argument similar to that above for x, $\alpha_r(x) \in F_{\alpha_s}$ for each $s \neq r$, and so $\alpha_s[\alpha_r(x)] = \alpha_r(x)$ whenever $s \neq r$ by Definition 2. Thus $\alpha_i(x) = x$ for $r+1 \le i \le k$ and $\alpha_i[\alpha_r(x)] = \alpha_r(x)$ for $1 \le i \le r-1$, so that $\alpha_{r+1} \cdots \alpha_k(x) = x$ and $\alpha_1 \cdots \alpha_{r-1}[\alpha_r(x)] = \alpha_r(x)$. Therefore $\alpha(x) = \alpha_1 \cdots \alpha_{r-1}\alpha_r\alpha_{r+1} \cdots \alpha_k(x) = \alpha_1 \cdots \alpha_{r-1}\alpha_r(x) = \alpha_1 \cdots \alpha_{r-1}[\alpha_r(x)] = \alpha_r(x) \neq x$ since $x \in T_{\alpha_r}$. Consequently $x \in T_\alpha$, and so $T_{\alpha_r} \subseteq T_\alpha$.

Similarly, if $1 \le j \le m$ then $\beta(x) = \beta_j(x) \neq x$ for each $x \in T_{\beta_j}$ and $T_{\beta_j} \subseteq T_\beta$, and so $T_{\alpha_r} \cap T_{\beta_j} \subseteq T_\alpha \cap T_\beta$. Thus if $x \in T_{\alpha_r} \cap T_{\beta_j}$, then $x \in T_\alpha \cap T_\beta$. Therefore $\alpha(x) = \beta(x)$ by Definition 4 since $\alpha$ and $\beta$ are semidisjoint. Hence $\alpha_r(x) = \alpha(x) = \beta(x) = \beta_j(x)$, and so $\alpha_r$ and $\beta_j$ are semidisjoint.

Since $\{\alpha_i\}_{i=1}^{k}$ and $\{\beta_i\}_{i=1}^{m}$ are each disjoint collections in Sym(S), then $\{\alpha_i\}_{i=1}^{k}$ and $\{\beta_i\}_{i=1}^{m}$ are each semidisjoint collections in Sym(S) by Corollary 7.

Thus $\alpha_r$ and $\alpha_j$ are semidisjoint for each r and j ($1 \le r \le k$, $1 \le j \le k$) by Definition 4. Similarly, $\beta_r$ and $\beta_j$ are also semidisjoint for each r and j ($1 \le r \le m$, $1 \le j \le m$). Since it was established above that $\alpha_r$ and $\beta_j$ are semidisjoint for each r and j such that $1 \le r \le k$ and $1 \le j \le m$, then $\{\alpha_i\}_{i=1}^{k} \cup \{\beta_i\}_{i=1}^{m}$ is a semidisjoint collection in Sym(S) by Definition 4.

Lemma 10 cannot be extended to conclude either that $\alpha_r$ and $\beta_j$ are disjoint for each r and j such that $1 \le r \le k$, $1 \le j \le m$ or that $\{\alpha_i\}_{i=1}^{k} \cup \{\beta_i\}_{i=1}^{m}$ is a disjoint collection of permutations in Sym(S). That is, there exist disjoint collections $\{\alpha_i\}_{i=1}^{k}$ and $\{\beta_i\}_{i=1}^{m}$ of distinct permutations for which $\alpha = \alpha_1 \cdots \alpha_k$ and $\beta = \beta_1 \cdots \beta_m$ are semidisjoint, but for which $\alpha_r$ and $\beta_j$ are not disjoint for some r and j, and $\{\alpha_i\}_{i=1}^{k} \cup \{\beta_i\}_{i=1}^{m}$ is not a disjoint collection of permutations in Sym(S).

Consider the example following Lemma 9 in which $\alpha, \beta \in S_4$ are defined by $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$. By Definition 3, each of $\{\alpha\}$ and $\{\beta\}$ is a disjoint collection of distinct permutations in $S_4$. Furthermore, $\alpha$ and $\beta$ are trivially the products of the permutations in $\{\alpha\}$ and $\{\beta\}$, respectively. Finally, it was shown above that $\alpha$ and $\beta$ are semidisjoint. However, since $T_\alpha = \{1,2\}$ and $T_\beta = \{1,2,3,4\}$, then $T_\alpha \cap T_\beta = \{1,2\} \ne \varnothing$, so that $\alpha$ and $\beta$ are not disjoint by Definition 3. Moreover, since it was argued above that $\alpha \ne \beta$, then $\{\alpha\} \cup \{\beta\} = \{\alpha, \beta\}$ is not a disjoint collection of permutations in $S_4$ according to Definition 3. A more substantial example is provided in the concluding remarks.

However, a weak form of Lemma 10 can be extended in the manner described above. For if the factors $\{\alpha_i\}_{i=1}^{k}$ and $\{\beta_i\}_{i=1}^{m}$ of $\alpha$ and $\beta$, respectively, are each disjoint collections of distinct cycles in Sym(S) rather than general permutations, then $\{\alpha_i\}_{i=1}^{k} \cup \{\beta_i\}_{i=1}^{m}$ is indeed a disjoint collection in Sym(S).

**Corollary 11**: Suppose $\alpha, \beta \in$ Sym(S), $\alpha$ and $\beta$ are semidisjoint, $\alpha = \alpha_1 \cdots \alpha_k$, $\beta = \beta_1 \cdots \beta_m$, and each of $\{\alpha_i\}_{i=1}^{k}$ and $\{\beta_i\}_{i=1}^{m}$ is a disjoint collection of distinct cycles in Sym(S). Then $\{\alpha_i\}_{i=1}^{k} \cup \{\beta_i\}_{i=1}^{m}$ is a disjoint collection of cycles in Sym(S).

Proof: If $1 \le r \le k$ and $1 \le j \le m$, then $\alpha_r$ and $\beta_j$ are semidisjoint by Lemma 10. Thus if $\alpha_r \ne \beta_j$, then $\alpha_r$ and $\beta_j$ are disjoint by Lemma 9 since $\alpha_r$ and $\beta_j$ are cycles. Since $\{\alpha_i\}_{i=1}^{k}$ and $\{\beta_i\}_{i=1}^{m}$ are each disjoint collections of distinct cycles in Sym(S), then $\{\alpha_i\}_{i=1}^{k} \cup \{\beta_i\}_{i=1}^{m}$ is also a disjoint collection of cycles in Sym(S).

**Main Results**

We are now prepared to present the main results on commutativity for semidisjoint permutations. These two results correspond to somewhat different special cases of the most general commutativity results obtained for disjoint permutations ([7, Theorem 9],[7, Corollary 11]).

Recall that a weak version of [7, Theorem 9] restricts the permutations to cycles, stating that disjoint cycles in Sym(S) commute [6, p. 79, no. 10]. Part (a) of Theorem 12 verifies that the same is true for semidisjoint cycles in Sym(S). More specifically, Theorem 12(a) establishes that semidisjoint pairs of cycles on an arbitrary nonempty set S commute.

Furthermore, the main commutativity result for disjoint pairs of permutations in Sym(S) [7, Theorem 9] was extended to disjoint collections of permutations in Sym(S) [7, Corollary 11]. In a similar manner, the result of Theorem 12(a) for semidisjoint pairs of cycles in Sym(S) is easily extended to semidisjoint collections of cycles in Sym(S) in part (b) of Theorem 12.



**Theorem 12**: (Winton's First Theorem)

(a) If $\alpha$ and $\beta$ are semidisjoint cycles in Sym(S), then $\alpha\beta = \beta\alpha$.

(b) If C is a semidisjoint collection of cycles in Sym(S), then $\alpha\beta = \beta\alpha$ for each $\alpha,\beta \in C$.

Proof:

(a) If $\alpha$ and $\beta$ are semidisjoint cycles in Sym(S), then according to Lemma 9 either $\alpha = \beta$ or $\alpha$ and $\beta$ are disjoint. If $\alpha = \beta$, then clearly $\alpha\beta = \beta\alpha$. Otherwise $\alpha$ and $\beta$ are disjoint, and so $\alpha\beta = \beta\alpha$ [7, Theorem 9].

(b) The result follows immediately from Definition 4 and Theorem 12(a).

Now recall that another weak version of [7, Theorem 9] limits the result to a finite nonempty underlying set while leaving the permutations arbitrary, stating that general disjoint permutations in $S_n$ on a finite nonempty set commute [5, p. 47]. Part (a) of Theorem 13 verifies that the same is true for semidisjoint permutations in $S_n$ . In other words, Theorem 13(a) establishes that for a finite nonempty set S of order n, semidisjoint permutations in $S_n$ commute.

As noted above, the result that disjoint pairs of permutations in Sym(S) commute [7, Theorem 9] was extended to disjoint collections of permutations [7, Corollary 11]. Furthermore, the fact established in Theorem 12(a) that semidisjoint pairs of cycles in Sym(S) commute was extended to include semidisjoint collections of cycles in Theorem 12(b). Similarly, part (b) of

Theorem 13 extends the result in Theorem 13(a) for semidisjoint pairs of permutations in $S_n$ to semidisjoint collections of permutations in $S_n$.

**Theorem 13**: (Winton's Second Theorem)

(a) If $\alpha$ and $\beta$ are semidisjoint permutations in $S_n$, then $\alpha\beta = \beta\alpha$.

(b) If C is a semidisjoint collection of permutations in $S_n$, then $\alpha\beta = \beta\alpha$ for each $\alpha,\beta \in C$.

Proof:

(a) Suppose $\alpha$ and $\beta$ are semidisjoint permutations in $S_n$. Since $\alpha,\beta \in S_n$, then $\alpha$ and $\beta$ can each be written as finite products $\alpha = \alpha_1 \cdots \alpha_k$ and $\beta = \beta_1 \cdots \beta_m$ of distinct, pairwise disjoint nontrivial cycles ([1, p. 96, Theorem 2.21],[4, p. 133, Theorem 3.2.2]). Consequently each of $\{\alpha_i\}_{i=1}^{k}$ and $\{\beta_i\}_{i=1}^{m}$ is a disjoint collection of distinct cycles in $S_n$. Since $\alpha$ and $\beta$ are semidisjoint, then $\{\alpha_i\}_{i=1}^{k} \cup \{\beta_i\}_{i=1}^{m}$ is a disjoint collection of cycles in $S_n$ by Corollary 11. Therefore $\lambda\rho = \rho\lambda$ for each $\lambda,\rho \in \{\alpha_i\}_{i=1}^{k} \cup \{\beta_i\}_{i=1}^{m}$ [7, Corollary 11]. Hence

$\alpha\beta = \alpha_1 \cdots \alpha_k\, \beta_1 \cdots \beta_m = \beta_1\alpha_1 \cdots \alpha_k\beta_2 \cdots \beta_m = \beta_1\beta_2\alpha_1 \cdots \alpha_k\beta_3 \cdots \beta_m = \cdots = \beta_1 \cdots \beta_m\, \alpha_1 \cdots \alpha_k = \beta\alpha$.

(b) The result follows immediately from Definition 4 and Theorem 13(a).

**Concluding Remarks**

As stated above, the most basic result related to commutativity in permutation groups commonly presented in literature is that disjoint cycles in $S_n$ commute. This result is limited both in its restriction to cycles and its restriction to a finite underlying set. However, this result has been generalized in both aspects by extending the result for disjoint cycles in $S_n$ to disjoint permutations in Sym(S) on an arbitrary nonempty set S [7, Theorem 9]. Theorem 12(a) extends the same basic result for disjoint cycles in $S_n$ to semidisjoint cycles in Sym(S). Theorem 13(a) provides a different generalization by extending the same basic result for disjoint cycles in $S_n$ to semidisjoint permutations in $S_n$. In the transition from Theorem 12 to Theorem 13, something is gained as well as lost. The result in Theorem 12 for semidisjoint cycles is extended to general semidisjoint permutations in Theorem 13. However, the arbitrary nonempty underlying set S in Theorem 12 is restricted to a finite nonempty set in Theorem 13.

A natural question that arises at this point is whether or not the results in Theorem 12 and Theorem 13 can be combined to produce a more general result corresponding to the one previously established for disjoint permutations in Sym(S) [7, Theorem 9] by showing that general semidisjoint permutations in

Sym(S) on an arbitrary nonempty set S commute. Unfortunately this goal cannot be achieved, as illustrated by the following counterexample.

In the Cartesian plane $\mathbf{R}^2$, define A = $\{(n,0) \mid n \text{ is an integer}; n \le -1\}$, B = $\{(0,n) \mid n \text{ is an integer}; n \le -1\}$, and C = $\{(0,n) \mid n \text{ is an integer}; n \ge 0\}$. Define $\alpha: \mathbf{R}^2 \to \mathbf{R}^2$ by $\alpha(n,0) = (n+1,0)$ for each $(n,0) \in A$; $\alpha(0,n) = (0,n+1)$ for each $(0,n) \in C$; $\alpha(x,y) = (x,y)$ for each $(x,y) \in \mathbf{R}^2 - (A \cup C)$. Similarly, define $\beta: \mathbf{R}^2 \to \mathbf{R}^2$ by $\beta(0,n) = (0,n+1)$ for each $(0,n) \in B \cup C$; $\beta(x,y) = (x,y)$ for each $(x,y) \in \mathbf{R}^2 - (B \cup C)$. Therefore $\alpha, \beta \in \text{Sym}(\mathbf{R}^2)$. (See Figure 1 below.)

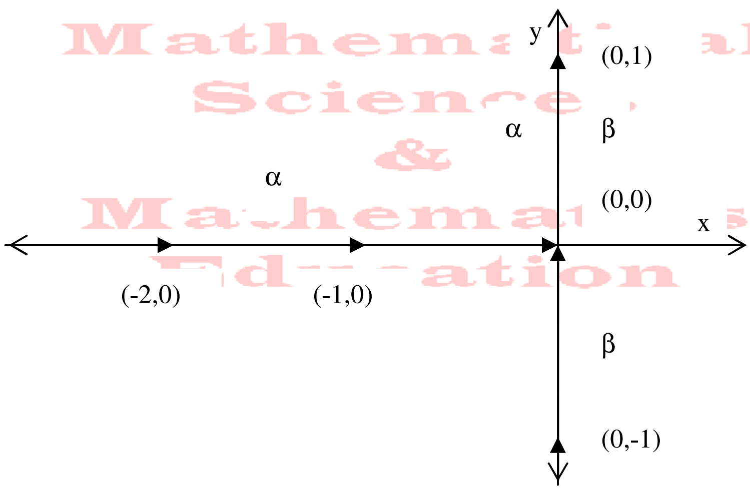


**Figure 1**

Then $T_\alpha = A \cup C$ and $T_\beta = B \cup C$, so that $T_\alpha \cap T_\beta = C$. Furthermore, if $(0,n) \in T_\alpha \cap T_\beta = C$ then $\alpha(0,n) = (0,n+1) = \beta(0,n)$. Therefore $\alpha$ and $\beta$ are semidisjoint by Definition 4.

Since $(-1,0) \in A$ then $\alpha(-1,0) = (0,0)$. However, $(-1,0) \notin B \cup C$, so that $\beta(-1,0) = (-1,0)$. Furthermore, since $(0,0) \in C$ then $\beta(0,0) = (0,1)$. Therefore $\alpha\beta(-1,0) = \alpha[\beta(-1,0)] = \alpha(-1,0) = (0,0)$. On the other hand, $\beta\alpha(-1,0) = \beta[\alpha(-1,0)] = \beta(0,0) = (0,1)$. Thus $\alpha\beta(-1,0) \ne \beta\alpha(-1,0)$, and so $\alpha\beta \ne \beta\alpha$.

Hence $\alpha$ and $\beta$ are semidisjoint permutations in $\text{Sym}(\mathbf{R}^2)$ with the property that $\alpha\beta \ne \beta\alpha$. Consequently, Theorem 12 and Theorem 13 cannot be combined and extended to show that general semidisjoint permutations on an arbitrary nonempty set S commute. Thus the corresponding result for disjoint permutations [7, Theorem 9] is not valid for semidisjoint permutations.

Additionally, since $T_\alpha \cap T_\beta = C \ne \varnothing$ then $\alpha$ and $\beta$ are not disjoint according to Definition 3. Furthermore, it is clear that $\alpha \ne \beta$ since, for example,

$\alpha(-1,0) = (0,0)$ but $\beta(-1,0) = (-1,0)$. Therefore $\alpha$ and $\beta$ are distinct permutations in Sym( $\mathbf{R}^2$ ) which are semidisjoint but not disjoint. Referring to the comments following Lemma 9, $\alpha$ and $\beta$ provide an additional example that the result in Lemma 9 for semidisjoint cycles in Sym(S) cannot be extended to arbitrary semidisjoint permutations in Sym(S).

Lastly, it was shown above that $\alpha$ and $\beta$ are semidisjoint. Furthermore, $\alpha$ and $\beta$ are trivially the products of the permutations in $\{\alpha\}$ and $\{\beta\}$, respectively. Finally, each of $\{\alpha\}$ and $\{\beta\}$ is clearly a (singleton) disjoint collection of distinct permutations in Sym( $\mathbf{R}^2$ ) by Definition 3. However, it was also shown above that $\alpha \neq \beta$ and that $\alpha$ and $\beta$ are not disjoint. Therefore $\{\alpha\} \cup \{\beta\} = \{\alpha,\beta\}$ is not a disjoint collection in Sym( $\mathbf{R}^2$ ) according to Definition 3. Referring to the comments following Lemma 10, $\alpha$ and $\beta$ provide another example that the result in Lemma 10 cannot be extended to conclude either that $\alpha_r$ and $\beta_j$ are disjoint for each r and j such that $1 \leq r \leq k$, $1 \leq j \leq m$ or that $\{\alpha_i\}_{i=1}^{k} \cup \{\beta_i\}_{i=1}^{m}$ is a disjoint collection of permutations in Sym(S).

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