Ore extension over α-quasi-Baer and α-p.q.-Baer rings

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Abstract

In this paper we extend some well known results of quasi-Baer and p.q.-Baer to α-quasi-Baer and α-p.q.-Baer using α-weakly rigid ring. Further, we investigate some results for quasi α-Armendariz ring and also we give some Examples to illustrate our theory.

Introduction

Throughout this paper R denotes an associative ring with identity, α is an endomorphism of R and δ an α-derivation of R, that is δ is an additive map such that $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$, for all $a, b \in R$. Kaplansky [11] introduced Baer rings to abstract various prospects of AW^*-Algebra and von-Neumann Algebra. Quasi-Baer rings (i.e. rings in which the right annihilator of every ideal is generated by an idempotent) introduced by Clark [5], are used to characterize when finite dimensional algebra with unity over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra. The definition of quasi-Baer ring is left-right i.e. a ring R is left (quasi) Baer if and only if R is right (qausi) Baer.

As a generalization of quasi-Baer ring, G.F. Birkenmeier, J.Y. Kim, and J.K. Park [4] introduced the concept of principally quasi-Baer rings. A ring R is called principally quasi-Baer (or right p.q.-Baer) if the right annihilator of a principal right ideal of R is generated by an idempotent. The class of p.q.-Baer ring includes all Baer rings, quasi-Baer rings, abelian p.p. rings and bi-regular rings. Further a number of authors investigated quasi-Baer and p.q.-Baer properties on different structures of a ring.

According to Krempa [12], a monomorphism α of a ring R is called to be rigid if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. A ring R is said to be α-rigid if there exists a rigid monomorphism α of R. Nasr-Isfahani et al. [14] generalized α-rigid ring to α-weakly rigid ring and used it to transfer the quasi-Baer property and p.q.-Baer property of an α-weakly rigid ring R to its extensions such as the skew polynomial ring $R[x;\alpha,\delta]$, skew Laurent polynomial ring $R[x,x^{-1};\alpha]$, skew power series ring $R[[x;\alpha]]$ and skew Laurent power series ring $R[[x,x^{-1};\alpha]]$.

A subset S of a ring R is called α-set if S is a α-stable set, i.e. $\alpha(S)\subseteq S$. α-Baer ring was defined by Han [6] as a ring in which the right annihilator of every α-set (resp. α-ideal) is generated by an idempotent is called α-Baer ring (resp. α-quasi-Baer ring). Also a ring R is called right (or left) α-p.q.-Baer (resp. right or left p.p.-ring) if the right (or left) annihilator of every right (or left) principal α-ideal (resp. α-element) is generated by an idempotent. R is called α-p.q.-Baer ring (resp. right or left p.p. ring) if it is both right α-p.q.-Baer and left α-p.q.-Baer. In [6] Han defined and analyzed the behavior of skew polynomial ring over the above mentioned properties for α-rigid ring.

In the present article we study the α-quasi-Baerness and α-p.q.-Baerness for an α-weakly rigid ring R and some of its extensions such as skew polynomial ring $R[x;\alpha]$, Ore extension $R[x;\alpha,\delta]$, skew power series ring $R[[x;\alpha]]$ and find some connectedness between an α-weakly rigid ring R and its extensions through some results which are a generalization of the results provided in [6], [14]. Further, we also show the same results for a quasi α-Armendariz ring R.

Ore extension with $\delta=1$ over α-quasi-Baer and α-p.q. -Baer ring

In this section we extend the results of [6] to α-weakly rigid ring. Further we define the notion of quasi α-Armendariz ring as a generalization of quasi-Armendariz ring. Also we prove the same results for quasi α-Armendariz ring. Recall from [14] a ring R is called α-weakly rigid if for each $a,b\in R, a\alpha(Rb)=0$ if and only if $aRb=0$. α-weakly rigid ring is a generalization of α-rigid ring and α-compatible ring. Now we give some examples to show that an α-weakly rigid ring R need not to be α-rigid.

Example 2.1 *Let Q be a ring of rational numbers then $M_2(Q)$ is a prime ring. Suppose α be an automorphism of $M_2(Q)$ which is defined as follows:*

$$\alpha\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

for each $a,b,c,d\in Q$. Since $M_2(Q)$ is a prime ring and α is an automorphism of $M_2(Q)$, so $M_2(Q)$ is α-weakly rigid ring [14, Example

2.14]. Now we check that this ring is α-rigid or not. Take $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in M_2(Q)$, then

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \alpha\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

but $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \neq 0$. Thus $M_2(Q)$ is not α-rigid.

Example 2.2 *We consider a ring* R *as*

$$R = \left\{ \begin{pmatrix} a & t \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z}, t \in Q \right\}$$

where $\mathbb{Z}$ and Q are the set of all integers and set of all rational numbers, respectively. Then R is a commutative ring. Let $\alpha : R \to R$ be an automorphism of R defined by

$$\alpha\left(\begin{pmatrix} a & t \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & t/2 \\ 0 & a \end{pmatrix}.$$

By [8, Theorem 1], R is not α-rigid ring. Now suppose any arbitrary $\begin{pmatrix} a & p \\ 0 & a \end{pmatrix}, \begin{pmatrix} b & q \\ 0 & b \end{pmatrix}$ and $\begin{pmatrix} c & r \\ 0 & c \end{pmatrix} \in R$ such that

$$\begin{pmatrix} a & t \\ 0 & a \end{pmatrix} \alpha\left(\begin{pmatrix} b & q \\ 0 & b \end{pmatrix}\begin{pmatrix} c & r \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

It follows that

$$\begin{pmatrix} abc & \dfrac{abr + acq}{2} + bcp \\ 0 & abc \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This gives $abc = 0$ and $\dfrac{abr + acq}{2} + bcp = 0$, which leads to the following:

1. $a = 0$ and $b = 0$
2. $a = 0$ and $c = 0$

3. $b = 0$ and $c = 0$
4. $a = 0 = b = c$

By considering that either of the above cases hold valid and true we find

$$\begin{pmatrix} a & t \\ 0 & a \end{pmatrix}\begin{pmatrix} b & q \\ 0 & b \end{pmatrix}\begin{pmatrix} c & r \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus R is α-weakly rigid ring.

For a ring R with an endomorphism α, there exists an endomorphism of $R[x;\alpha]$ which extends α. For example, consider a map $\overline{\alpha}$ on $R[x;\alpha]$ defined by $\overline{\alpha}(f(x)) = \alpha(a_0) + \alpha(a_1)x + \ldots + \alpha(a_n)x^n$ for all $f(x) = a_0 + a_1x + \ldots + a_nx^n \in R[x;\alpha]$. Then $\overline{\alpha}$ is an endomorphism of $R[x;\alpha]$ and $\overline{\alpha}(a) = \alpha(a)$ for all $a \in R$, that means $\overline{\alpha}$ is an extension of α. $\overline{\alpha}$ is called the extended endomorphism of α. Here, we shall denote the extended map $\overline{\alpha} : R[x;\alpha] \to R[x;\alpha]$ by α.

Now, we begin our main results with Theorem 2.2 but first we give Lemma 2.1 which is required to prove the main Theorems.

Lemma 2.1 *Let R be an α-weakly rigid ring and $e \in R$ a right semicentral idempotent of R. Then for positive integer m,* $e\alpha^m(r) = e\alpha^m(re)$.

Proof. Since e be a right semicentral idempotent of R so $eR = eRe$ which implies $er(1-e) = 0$ for all $e \in R$. It follows that $e\alpha^m(r(1-e)) = 0$ since R be an α-weakly rigid ring. Thus $e\alpha^m(r) = e\alpha^m(re)$.

Theorem 2.2 *Let R be an α-weakly rigid ring. Then the following conditions are equivalent:*

1. R is an α-quasi-Baer ring;
2. $R[x;\alpha]$ is a quasi-Baer ring;
3. $R[x;\alpha]$ is an α-quasi-Baer ring for every extended α-automorphism of $R[x;\alpha]$.

Proof. $(1) \Rightarrow (2)$ Suppose R is α-quasi-Baer and I be an arbitrary ideal of $R[x;\alpha]$. Consider the set I_0 of all the leading coefficients of elements in I i.e. $I_0 = \{a_n \in R \mid f(x) = a_n x^n + \sum_{i=0}^{n-1} a_i x^i \in I\}$. Then I_0 is an ideal of R. Note that I_0 is an α-ideal of R, since for $f(x) = a_n x^n + \sum_{i=0}^{n-1} a_i x^i \in I$ and $g(x) = x \in R$, we have $g(x)f(x) = \alpha(a_n)x^{n+1} + \sum_{i=0}^{n-1} \alpha(a_i)x^{i+1} \in I$ and so $\alpha(a_n) \in I_0$. Thus I_0 is an α-ideal of R. Since R is α-quasi-Baer, $l_R(I_0) = Re$ for some right semicentral idempotent $e \in R$ which gives $ea_n = 0$ for all $a_n \in I_0$. Now we show $R[x;\alpha]e = l_{R[x;\alpha]}(I)$. For any $f(x) = a_n x^n + \sum_{i=0}^{n-1} a_i x^i \in I$ we have $a_n \in I_0$, so $ea_n = 0$. Therefore $ef(x) = e(\sum_{i=0}^{n-1} a_i x^i) = ea_{n-1}x^{n-1} + \sum_{i=0}^{n-2} a_i x^i$. Since $ea_{n-1} \in I_0$, we get $ea_{n-1} = eea_{n-1} = 0$. Continuing this way we get $ef(x) = 0$ and so $R[x;\alpha]e \subseteq l_{R[x;\alpha]}(I)$. Suppose $g(x) = \sum_{j=0}^{m} b_j x^j \in l_{R[x;\alpha]}(I)$, so for each $f(x) = \sum_{i=0}^{n} a_i x^i \in I$ and $r \in R$, $g(x)rf(x) = 0$. Therefore $b_m \alpha^m(ra_n) = 0$ for each $r \in R$ which implies that $b_m Ra_n = 0$ since R is α-weakly rigid ring, so $b_m = b_m e$. Now $g(x)rf(x) = b_m x^m rf(x) + \sum_{j=0}^{m-1} b_j x^j rf(x) = 0$. It follows that $\sum_{j=0}^{m-1} b_j x^j rf(x) = 0$ since $b_m x^m rf(x) = b_m ex^m rf(x) = b_m e\alpha^m(e)x^m rf(x) = b_m ex^m erf(x) = 0$ by Lemma 2.1. In the same way we find that $b_{m-1} = b_{m-1}e$. Continuing this way we get for each j $b_j = b_j e$, so $g = ge$ which gives $l_{R[x;\alpha]}(I) \subseteq R[x;\alpha]e$. Hence $R[x;\alpha]$ is quasi-Baer.

$(2) \Rightarrow (3)$ It is clear.

$(3) \Rightarrow (1)$ Suppose $R[x;\alpha]$ is α-quasi-Baer and α-weakly rigid. Let I be any α-ideal of R. Then by [6, Lemma 1.7] $R[x;\alpha]I$ is an α-ideal of $R[x;\alpha]$. Since $R[x;\alpha]$ is α-quasi-Baer, $l_{R[x;\alpha]}(R[x;\alpha]I) = R[x;\alpha]e$ for some idempotent $e(x) = \sum_{i=0}^{n} \in R[x;\alpha]$. Thus by [14, Lemma 3.5] $l_{R[x;\alpha]}(R[x;\alpha]I) = l_R(I)[x;\alpha] = R[x;\alpha]e(x)$ which implies that $e_i \in l_R(I)$ for all i. Again by [14, Lemma 3.5]

$l_R(I) = l_R(I)[x;\alpha] \cap R = R[x;\alpha]e(x) \cap R$ it follows that for any $a \in l_R(I)$, $a = ae_0$. Hence, R is α-quasi-Baer.

Corollary 2.3 ([Theorem 2.3]6) *Let R be a ring with an endomorphism α and let Λ_α be the set of all extended endomorphisms on $R[x;\alpha]$ of α. If R is α-rigid, then the following are equivalent:*

1. R is α-quasi-Baer;
2. $R[x;\alpha]$ is quasi-Baer;
3. $R[x;\alpha]$ is $\overline{\alpha}$-quasi-Baer for all $\overline{\alpha} \in \Lambda_\alpha$.

Theorem 2.4 *Let R be an α-weakly rigid ring. Then the following conditions are equivalent:*

1. R is left α-p.q.-Baer;
2. $R[x;\alpha]$ is a left p.q.-Baer ring;
3. $R[x;\alpha]$ is a left α-p.q.-Baer ring for every extended α-automorphism of $R[x;\alpha]$.

Proof. $(1) \Rightarrow (2)$ Let R be α-weakly rigid left α-p.q.-Baer ring and I be a left principal ideal of $R[x;\alpha]$ which is generated by $h(x) = \sum_{i=0}^{n} h_i x^i \in R[x;\alpha]$ i.e. $I = \{ f(x)h(x) \mid f(x) \in R[x;\alpha] \}$. Note that I_0 is a left ideal of R which is generated by $h_0, h_1, \ldots, h_n$ i.e. $I_0 = \{ rh_i \mid r \in R \}$. Take $g(x) = x$, $g(x)Rh(x) = xRh(x) = \sum_{i=0}^{n} \alpha(Rh_i)x^{i+1}$ and so $\alpha(Rh_i) \in I_0$ for each i. Thus I_0 is an left principal α-ideal of R. Since R is α-p.q.-Baer, $l_R(Rh_i) = Re_i$ where e_i be right semicentral idempotents of R therefore $e_i Rh_i = 0$ for all i. Let $e = e_0 e_1 \ldots e_n$ which implies e is also a right semicentral idempotent of R. Thus by [14 ,Corollary 3.3] e is a right semicentral idempotent of $R[x;\alpha]$. We show that $l_{R[x;\alpha]}(R[x;\alpha]h(x)) = R[x;\alpha]e$. For $h(x) = \sum_{i=0}^{n} h_i x^i \in R[x;\alpha]$, $eh(x) = 0$ which implies $ef(x)h(x) = ef(x)eh(x) = 0$ for any $f(x) \in R[x;\alpha]$. Therefore $R[x;\alpha]e \subseteq l_{R[x;\alpha]}(R[x;\alpha]h(x))$. Again suppose any $f(x) = \sum_{j=0}^{m} a_j x^j \in l_{R[x;\alpha]}(R[x;\alpha]h(x))$. Then

$f(x)R[x;\alpha]h(x)=0$ which implies $f(x)Rh(x)=0$ it follows that $a_j r h_i = 0$ for all $r \in R$ from [14, Theorem 3.9]. Thus $a_j \in l_R(Rh_i) = Re_i$ which gives $a_j = a_j e$ so $f = fe$ and therefore $l_{R[x;\alpha]}(R[x;\alpha]h(x)) = R[x;\alpha]e$. Hence $R[x;\alpha]$ is left p.q.-Baer.

$(2) \Rightarrow (3)$ It is straightforward.

$(3) \Rightarrow (1)$ Similar to Theorem 2.2

The concept of α-skew Armendariz ring has been introduced in [8] which is a generalization of α-rigid ring and α-Armendariz ring. A ring R is said to be α-skew Armendariz ring, if for $p = \sum_{i=0}^{m} a_i x^i$ and $q = \sum_{j=0}^{n} b_j x^j$ in $R[x;\alpha]$ the condition $pq = 0$ implies $a_i \alpha^i(b_j) = 0$ for all i and j.

The Armendariz property of rings was extended to skew polynomial rings in [10]. Following Hong et al [10], a ring R is called α-Armendariz if for $p = \sum_{i=0}^{m} a_i x^i$ and $q = \sum_{j=0}^{n} b_j x^j$ in $R[x;\alpha]$ the condition $pq = 0$ implies $a_i b_j = 0$ for all i and j. α-Armendariz ring is a generalization of α-rigid ring and Armendariz ring. Hong et al [10] proved that an α-Armendariz ring is α-skew Armendariz.

In [3] Baser and Kwak introduced the concept of α-quasi Armendariz ring. A ring R is called quasi-Armendariz ring with the endomorphism α (or simply α-quasi Armendariz) if for $p(x) = a_0 + a_1 x + \ldots + a_m x^m, q(x) = b_0 + b_1 x + \ldots + b_n x^n$ in $R[x;\alpha]$ satisfy $p(x)R[x;\alpha]q(x) = 0$, implies $a_i R[x;\alpha] b_j = 0$ for all $0 \le i \le m$ and $0 \le j \le n$ or equivalently, $a_i R \alpha^t(b_j) = 0$ for any nonnegative integer t and all i, j. Baser and Kwak [3] also showed that every α-quasi Armendariz ring is α-skew quasi Armendariz in case that α is an epimorphism, but the converse does not hold, in general. Motivated by [3], Pourtaherian and Rakhimov [15] introduced quasi α-Armendariz ring which is a generalization of quasi-Armendariz ring. A ring R is called a quasi α-Armendariz (or simply q. α-Armendariz) ring if whenever $p = \sum_{i=0}^{m} a_i x^i$ and $q = \sum_{j=0}^{n} b_j x^j$ in $R[x;\alpha]$ satisfy $pR[x;\alpha]q = 0$, we have $a_i R b_j = 0$ for all i and j. It is easy to see that an α-rigid ring is quasi α-Armendariz.

Here we refer to an Example from [15] that describes about a quasi α-Armendariz ring which is not α-rigid.

Given a ring R and a bimodule $_RM_R$. The trivial extension of R by M is the ring $T(R,M)=R\bigoplus M$ with the usual addition and multiplication defined as follows:

$$(r_1,m_1)(r_2,m_2)=(r_1r_2,r_1m_2+m_1r_2).$$

Example 2.3 (15, Example 3) *Let $R=T(\mathbb{Z},\mathbb{Q})$ be the trivial extension of $\mathbb{Z}$ by $\mathbb{Q}$, with automorphism $\alpha:R\to R$ defined by $\alpha((a,s))=(a,s/2)$. The ring R is quasi α-Armendariz but is not α-rigid.*

Now we show that a quasi Armendariz ring need not be an α-weakly rigid ring through the following example.

Example 2.4 *Let $R=\{(a,b)\in\mathbb{Z}\bigoplus\mathbb{Z}\mid a\equiv b(mod\,2)\}$ be a ring which is commutative reduced ring [9, Example 9], so R is a semiprime ring. Thus R is quasi Armendariz but not α-rigid. Now we check that R is an α-weakly rigid ring or not. Define $\alpha:R\to R$, such that $\alpha((a,b))=(b,a)$ an automorphism of R. Let $(0,2),(2,0)\in R$, then*

$$(0,2)\alpha((2,0)(2,0))=(0,2)\alpha(4,0)=0,$$

while

$$(0,2)(2,0)(2,0)=(0,8)\neq 0,$$

Thus it is clear that R is not α-weakly rigid ring.

To prove the results for a quasi α-Armendariz ring we need to construct Lemma which is given as follows

Lemma 2.5 *Let R be a quasi α-Armendariz ring, then the following conditions hold:*

1. If $arb=0$ then $\alpha^n(a)rb=0$.
2. If $a\alpha^m(rb)=0$ then $arb=0$.

where $a,r,b\in R$ and m,n be some positive integers.

Proof. (1) Suppose $arb=0$ and $f(x)=\alpha(a)x, g(x)=bx\in R[x;\alpha]$. Then $f(x)rg(x)=(\alpha(a)x)r(bx)=\alpha(a)\alpha(rb)x^2=\alpha(arb)x^2=0$ which implies $\alpha(a)rb=0$ or $\alpha^n(a)rb=0$, since R is quasi α-Armendariz.

(2) Consider $a\alpha^m(rb)=0$ and $f(x)=ax^m, g(x)=bx\in R[x;\alpha]$ then $f(x)rg(x)=a\alpha^m(rb)x^{m+1}=0$. Thus $arb=0$ since R is quasi α-Armendariz.

Theorem 2.6 *Let R be a quasi α-Armendariz ring. If R is α-quasi-Baer ring then $R[x;\alpha]$ is a quasi-Baer ring.*

Proof. Let R be a quasi α-Armendariz and α-quasi-Baer ring, and I be any arbitrary ideal of $R[x;\alpha]$. Consider I_0 be the set of all the coefficients of elements of I. Observe that I_0 is an α-ideal of R since for $f(x)=\sum_{i=0}^{n}a_ix^i\in I$ and $g(x)=x\in R$, we have $g(x)f(x)=\sum_{i=0}^{n}\alpha(a_i)x^{i+1}\in I$ and so $\alpha(a_i)\in I_0$ for each i. Thus I_0 is an α-ideal of R, which gives $l_R(I_0)=Re$ for some right semicentral idempotent $e\in R$ i.e. $ea_i=0$ for any $a_i\in I_0$. Now we show that $l_{R[x;\alpha]}(I)=R[x;\alpha]e$. Suppose $f(x)=\sum_{i=0}^{n}a_ix^i\in I$, so $ef(x)=e(\sum_{i=0}^{n}a_ix^i)=\sum_{i=0}^{n}(ea_i)x^i=0$ so $R[x;\alpha]e\subseteq l_{R[x;\alpha]}(I)$. Again suppose $g(x)=\sum_{j=0}^{m}b_jx^j\in l_{R[x;\alpha]}(I)$ which implies $g(x)rf(x)=0$, it follows that $b_jra_i=0$ since R is quasi α-Armendariz. Then $b_j\in l_R(a_i)=Re$ which gives $b_j=b_je$. Thus $g=ge$ and therefore $l_{R[x;\alpha]}(I)\subseteq R[x;\alpha]e$. Hence $R[x;\alpha]$ is a quasi-Baer ring.

Theorem 2.7 *Let R be a quasi α-Armendariz ring. If R is left α-p.q.-Baer ring then $R[x;\alpha]$ is a left p.q.-Baer ring.*

Proof. Let R be quasi α-Armendariz left α-p.q.-Baer ring and I be a left principal ideal of $R[x;\alpha]$ which is generated by $h(x)=\sum_{i=0}^{n}h_ix^i\in R[x;\alpha]$ i.e. $I=\{f(x)h(x)\mid f(x)\in R[x;\alpha]\}$. Note that I_0 is a left ideal of R which is generated by $h_0,h_1,\ldots,h_n$ i.e. $I_0=\{rh_i\mid r\in R\}$. Take $g(x)=x$, $g(x)h(x)=xh(x)=\sum_{i=0}^{n}\alpha(h_i)x^{i+1}$ and so $\alpha(h_i)\in I_0$ for each i. Thus I_0 is an left principal α-ideal of R.

Since R is α-p.q.-Baer so $l_R(Rh_i) = Re_i$ where e_i be right semicentral idempotents of R. Let $e = e_0e_1 \ldots e_n$ which implies e is also a right semicentral idempotent of R. We show that $l_{R[x;\alpha]}(I) = R[x;\alpha]e$. For any $h(x) = \sum_{i=0}^{n} h_i x^i \in R[x;\alpha]$

$erh = e(\sum_{i=0}^{n}(rh_i)x^i) = \sum_{i=0}^{n}(e = e_0e_1 \ldots e_n)(rh_i)x^i$ which implies $erh = 0$. Thus $R[x;\alpha]e \subseteq l_{R[x;\alpha]}(R[x;\alpha]h(x))$. Again suppose any $f(x) = \sum_{j=0}^{m} a_j x^j \in l_{R[x;\alpha]}(R[x;\alpha]h(x))$. Then $f(x)R[x;\alpha]h(x) = 0$ which implies $f(x)Rh(x) = 0$ it follows that $a_j rh_i = 0$ for all $r \in R$. Thus $a_j \in l_R(Rh_i) = Re_i$ which gives $a_j = a_j e$ so $f = fe$ and therefore $l_{R[x;\alpha]}(R[x;\alpha]h(x)) = R[x;\alpha]e$. Hence $R[x;\alpha]$ is left p.q.-Baer.

Ore extension over α-quasi-Baer and α-p.q.-Baer ring

This section discusses about Ore extensions of α-quasi-Baer and α-p.q.-Baer rings. In [9] Hong et al. have shown that if R is an α-rigid ring, then R is Baer if and only if $R[x;\alpha,\delta]$ is a Baer ring. Nasr-Isfahani et al. [14] extended this result for α-weakly rigid ring to quasi-Baer and p.q.-Baer ring. Here we generalize these results to α-quasi-Baer and α-p.q.-Baer ring.

Recall from [13] an ideal I of a ring R with an automorphism α and an α-derivation δ is called an (α,δ)-ideal of R if $\alpha(I) = I$ and $\delta(I) \subseteq I$. A ring R with an automorphism α and an α-derivation δ is called an (α,δ)-quasi-Baer if the left annihilator of every (α,δ)-ideal is generated by an idempotent of R.

To prove the main results of this section we need the following Lemma which is a extension of [6, Lemma 1.1].

Lemma 3.1 *Let R be a ring with an automorphism α and an α-derivation δ. Then*

1. If I is a right (α,δ)-ideal of R, then RI is a right (α,δ)-ideal of R;
2. If I is a left (α,δ)-ideal of R, then IR is a left (α,δ)-ideal of R.

Proof. It follows from [6, Lemma 1.1].

Lemma 3.2 *Let R be a ring, α be an automorphism and δ an α-derivation of R with $\alpha\delta = \delta\alpha$. Then the following conditions hold:*

1. If I be an (α,δ)-ideal of R then $IR[x;\alpha,\delta]$ be an (α,δ)-ideal of $R[x;\alpha,\delta]$;
2. If I be a right principal (α,δ)-ideal of R then $IR[x;\alpha,\delta]$ be a right principal (α,δ)-ideal of $R[x;\alpha,\delta]$;
3. If I be a left principal (α,δ)-ideal of R then $R[x;\alpha,\delta]I$ be a left principal (α,δ)-ideal of $R[x;\alpha,\delta]$.

For a ring R with an automorphism α and α-derivation δ with $\alpha\delta = \delta\alpha$, there exists an α-derivation on $R[x;\alpha,\delta]$ which extends δ. For example, consider the automorphism $\overline{\alpha}$ and the $\overline{\alpha}$-derivation $\overline{\delta}$ on $R[x;\alpha],\delta$ defined by

$$\overline{\alpha}(f(x)) = \alpha(a_0) + \alpha(a_1)x + \ldots + \alpha(a_n)x^n$$

$$\overline{\delta}(f(x)) = \delta(a_0) + \delta(a_1)x + \ldots + \delta(a_n)x^n$$

for all $f(x) = a_0 + a_1x + \ldots + a_nx^n \in R[x;\alpha,\delta]$ and $\overline{\alpha}(r) = \alpha(r), \overline{\delta}(r) = \delta(r)$ for all $r \in R$. We shall denote the extended map $\overline{\alpha}: R[x;\alpha,\delta] \to R[x;\alpha,\delta]$ and $\overline{\delta}: R[x;\alpha,\delta] \to R[x;\alpha,\delta]$ by δ, and the image of $f \in R[x;\alpha,\delta]$ by $\alpha(f), \delta(f)$, respectively.

Theorem 3.3 *Let R be an (α,δ)-weakly rigid ring, α be an automorphism and δ an α-derivation of R with $\alpha\delta = \delta\alpha$. Then the following conditions are equivalent:*

1. R is an (α,δ)-quasi-Baer ring;
2. $R[x;\alpha,\delta]$ is an α-quasi-Baer ring;
3. $R[x;\alpha,\delta]$ is an (α,δ)-quasi-Baer ring for every extended α-derivation δ of $R[x;\alpha,\delta]$.

Proof. $(1) \Rightarrow (2)$ Let R be an α-weakly rigid and (α,δ)-quasi-Baer ring, and I be any α-ideal of $R[x;\alpha,\delta]$. Suppose that I_0 be an α-ideal of R which is a set of all the leading coefficients of polynomials in I i.e. $I_0 = \{a_n \in R \mid f(x) = a_nx^n + \sum_{i=0}^{n-1} a_ix^i \in I\}$. Now first we show that I_0 is a (α,δ)-ideal of R. Take any $g(x) = x \in R[x;\alpha,\delta]$ and

$f(x)=\sum_{i=0}^{n}a_i x^i \in I$,

$g(x)f(x)=xf(x)=\alpha(a_n)x^{n+1}+\sum_{i=0}^{n-1}\alpha(a_i)x^{i+1}+\delta(a_n)x^n+\sum_{i=0}^{n-1}\delta(a_i)x^i \in I$

. which gives $\delta(a_n)\in I_0$. Therefore I_0 is an (α,δ)-ideal of R . Since R is an (α,δ)-quasi-Baer ring so $l_R(I_0)=Re$ for any right semicentral idempotent $e\in R$ which implies $eI_0=0$. For $f(x)=\sum_{i=0}^{n}a_i x^i \in I$, $ef(x)=e\sum_{i=0}^{n}a_i x^i =0$. Thus $R[x;\alpha,\delta]e\subseteq l_{R[x;\alpha,\delta]}(I)$. Again suppose $g(x)=\sum_{j=0}^{m}b_j x^j \in l_{R[x;\alpha,\delta]}(I)$ then $g(x)rf(x)=0$ which implies $b_j r a_i =0$ since R is α-weakly rigid (from the proof of Theorem 2.2). Then $b_j\in l_R(a_i)=Re$, so $b_j=b_j e$ and thus $g=ge$. Therefore $l_{R[x;\alpha,\delta]}(I)\subseteq R[x;\alpha,\delta]e$. Hence $R[x;\alpha,\delta]$ is α-quasi-Baer.

$(2)\Rightarrow(3)$ It is straightforward.

$(3)\Rightarrow(1)$ Similar to Theorem 2.2.

Corollary 3.4 ([theorem 3.4]14) *Let R be an α-weakly rigid ring. If R is a quasi- Baer ring then $R[x;\alpha,\delta]$ is a quasi-Baer ring.*

Corollary 3.5 ([theorem 3.6]14) *Let R be an α-weakly rigid ring. If $R[x;\alpha,\delta]$ is a quasi-Baer ring then R is a quasi-Baer ring.*

Now we focus on extending the quasi α-Armendariz property of a skew polynomial rings, as described in section 2, to Ore extension.

Definition 3.6 *A ring R is called a quasi α-Armendariz ring if whenever $p=\sum_{i=0}^{m}a_i x^i$ and $q=\sum_{j=0}^{n}b_j x^j$ in $R[x;\alpha,\delta]$ satisfy $pR[x;\alpha,\delta]q=0$, we have $a_iRb_j=0$ for all i and j.*

Here, we introduce the concept of (α,δ)-p.q. Baer ring which is a generalization of α-quasi-Baer, (α,δ)-quasi-Baer and α-p.q.Baer by the following definition:

Definition 3.7 *A ring R with an automorphism α and an α-derivation δ is called an (α,δ)-p.q.-Baer if the left annihilator of every left principal (α,δ)-ideal is generated by an idempotent of R.*

Theorem 3.8 *Let R be an (α,δ)-weakly rigid ring, α be an automorphism and δ an α-derivation of R with $\alpha\delta=\delta\alpha$. Then the following conditions are equivalent:*

1. R is an (α,δ)-p.q.-Baer ring;
2. $R[x;\alpha,\delta]$ is an α-p.q.-Baer ring;
3. $R[x;\alpha,\delta]$ is an (α,δ)-p.q.-Baer ring for every extended α-derivation δ of $R[x;\alpha,\delta]$.

Proof. $(1)\Rightarrow(2)$ Let R be an α-weakly rigid (α,δ)-quasi-Baer ring and I be a left principal α-ideal of $R[x;\alpha,\delta]$ which is generated by $h(x)=\sum_{i=0}^{n}h_i x^i \in R[x;\alpha,\delta]$ i.e. $I=\{f(x)h(x) \mid f(x)\in R[x;\alpha,\delta]\}$. Note that I_0 is a left ideal of R which is generated by $h_0,h_1,\ldots,h_n$ i.e. $I_0=\{rh_i \mid r\in R\}$.

Take $g(x)=x$,

$g(x)rh(x)=xrh(x)=\sum_{i=0}^{n}\alpha(rh_i)x^{i+1}+\sum_{i=0}^{n}\delta(ra_i)x^i\in I$ and so $\delta(rh_i)\in I_0$ for each i. Thus I_0 is an left principal (α,δ)-ideal of R. The proof of the remaining part is similar to Theorem 2.3.

$(2)\Rightarrow(3)$ It is straightforward.

$(3)\Rightarrow(1)$ Similar to Theorem 2.3

Corollary 3.9 ([Theorem 3.9]14) *Let R be an α-weakly rigid ring. If R is a left p.q.-Baer ring then $R[x;\alpha,\delta]$ is a left p.q.-Baer ring.*

Corollary 3.10 ([Theorem 3.11]14) *Let R be an α-weakly rigid ring. If $R[x;\alpha,\delta]$ is a left p.q.-Baer ring then R is a left p.q.-Baer ring.*

Here, we show main results of this section using quasi (α,δ)-Armendariz ring in place of (α,δ)-weakly rigid ring. First, we define quasi (α,δ)-Armendariz ring which is an extension of quasi α-Armendariz ring.

Definition 3.11 *A ring R is called a quasi (α,δ)-Armendariz ring if whenever $p=\sum_{i=0}^{m}a_i x^i$ and $q=\sum_{j=0}^{n}b_j x^j$ in $R[x;\alpha,\delta]$ satisfy $pR[x;\alpha,\delta]q=0$, we have $a_iRb_j=0$ for all i and j.*

Theorem 3.12 *Let R be a quasi (α,δ)-Armendariz and (α,δ)-quasi-Baer ring then $R[x;\alpha,\delta]$ is a α-quasi-Baer ring.*

Proof. Let R be a quasi (α,δ)-Armendariz and (α,δ)-quasi-Baer ring, and I be any α-ideal of $R[x;\alpha,\delta]$. Suppose that I_0 be an α-ideal of R which is a collection of all the coefficients of elements of I. Now first we show that I_0 is a (α,δ)-ideal of R. Take any $g(x)=x\in R[x;\alpha,\delta]$ and $f(x)=\sum_{i=0}^{n}a_ix^i\in I$, $g(x)f(x)=xf(x)=\sum_{i=0}^{n}\alpha(a_i)x^{i+1}+\sum_{i=0}^{n}\delta(a_i)x^i\in I$. Thus $\sum_{i=0}^{n}\delta(a_i)x^i\in I$ since I is an α-ideal of $R[x;\alpha,\delta]$ so $\delta(a_i)\in I_0$. Therefore I_0 is an (α,δ)-ideal of R. Since R is an (α,δ)-quasi-Baer ring so $l_R(I_0)=Re$ for any right semicentral idempotent $e\in R$ which implies $eI_0=0$. Now to show $l_{R[x;\alpha,\delta]}(I)=R[x;\alpha,\delta]e$. Suppose $f(x)=\sum_{i=0}^{n}a_ix^i\in I$ so $ef=e\sum_{i=0}^{n}a_ix^i=0$. Thus $R[x;\alpha,\delta]e\subseteq l_{R[x;\alpha,\delta]}(I)$. Again suppose $g(x)=\sum_{j=0}^{m}b_jx^j\in l_{R[x;\alpha,\delta]}(I)$ then $g(x)rf(x)=0$ which implies $b_jra_i=0$ since R is quasi (α,δ)-Armendariz. Then $b_j\in l_R(a_i)=Re$ so $b_j=b_je$ and thus $g=ge$. Therefore $l_{R[x;\alpha,\delta]}(I)\subseteq R[x;\alpha,\delta]e$. Hence the result follows.

Theorem 3.13 *Let R be a quasi (α,δ)-Armendariz and left (α,δ)-p.q.-Baer ring then $R[x;\alpha,\delta]$ is a left α-p.q.-Baer.*

Proof. Suppose R is a quasi (α,δ)-Armendariz and left (α,δ)-p.q.-Baer ring, and I be any left α-ideal of $R[x;\alpha,\delta]$ which is generated by $h(x)=\sum_{i=0}^{n}h_ix^i\in R[x;\alpha,\delta]$ i.e. $I=\{f(x)h(x)\mid f(x)\in R[x;\alpha,\delta]\}$. Let I_0 be the set of all coefficients of elements of I. Then I_0 be a left α-ideal of R which is generated by $h_0,h_1,\ldots,h_n$. Note that I_0 is a left (α,δ)-ideal of R by Theorem 2.6. Since R is a left (α,δ)-p.q.-Baer ring, $l_R(Rh_i)=Re_i$ where e_i be semicentral idempotents of R which implies

$e_iRh_i = 0$. Let $e = e_0e_1\ldots e_n$ which implies e is also a semicentral idempotent of R. Now consider any $h(x) = \sum_{i=0}^{n} h_ix^i \in I$ so $erh(x) = \sum_{i=0}^{n} er(h_ix^i) = \sum_{i=0}^{n}(e_0e_1\ldots e_n)r(h_ix^i) = 0$, since e_i is a semicentral idempotent of R. Therefore $R[x;\alpha,\delta]e \subseteq l_{R[x;\alpha,\delta]}(R[x;\alpha,\delta]h(x))$. Again suppose $g(x) = \sum_{j=0}^{m} b_jx^j \in l_{R[x;\alpha,\delta]}(R[x;\alpha,\delta]h(x))$. Then $g(x)R[x;\alpha,\delta]h(x) = 0$ which implies that $g(x)Rh(x) = 0$. It follows that $b_jrh_i = 0$ since R is quasi α-Armendariz. Thus $b_j \in l_R(Rh_i) = Re_i$ which gives $b_j = b_je_0e_1\ldots e_n$ and therefore $g = ge$ implies $l_{R[x;\alpha,\delta]}(R[x;\alpha,\delta]h(x)) \subseteq R[x;\alpha,\delta]e$. Hence the result follows.

Skew power series over α-quasi-Baer ring

In this section we consider the relationship between the properties of being α-quasi-Baer of a ring R and of the skew power series ring $R[[x;\alpha]]$. Further we introduce the concept of quasi α-Armendariz of power series type which is an extension of quasi α-Armendariz ring and also an extension of skew α-Armendariz property of a ring R defined in [15].

Theorem 4.1 *Let R be an α-weakly rigid ring. Then the following conditions are equivalent:*

1. R is an α-quasi-Baer ring;
2. $R[[x;\alpha]]$ is a quasi-Baer ring;
3. $R[[x;\alpha]]$ is a α-quasi-Baer ring for every extended α-automorphism of $R[[x;\alpha]]$.

Proof. (1) $\Rightarrow$ (2) Suppose R is α-quasi-Baer and I be an arbitrary ideal of $R[[x;\alpha]]$. Let I_0 be the set of leading coefficients of elements in I i.e. $I_0 = \{a_n \in R \mid$ there exists $a_nx^n + \sum_{i=n+1}^{\infty} \in I$, for some non-negative integer n and $a_i \in R\}$. Then I_0 is an ideal of R. Note that I_0 is an α-ideal of R since for $f(x) = a_nx^n + \sum_{i=n+1}^{\infty} a_ix^i \in I$ and $g(x) = x \in R$, we have $g(x)f(x) = \alpha(a_n)x^n \sum_{i=n+1}^{\infty} \alpha(a_i)x^i \in I$ and so $\alpha(a_i) \in I_0$ for each i. Thus I_0 is an α-ideal of R, which gives $l_R(I_0) = Re$ for some idempotent

$e \in R$. For any $f(x)=\sum_{i=n}^{\infty} a_i x^i \in I$ we have $a_n \in I_0$, so $ea_n = 0$. Therefore $ef(x)=e(\sum_{i=0}^{\infty} a_i x^i) = ea_n x + e(\sum_{i=n+1}^{\infty} a_i x^i)$. Since $ea_n \in I_0$, we get $ea_n = eea_n = 0$. Continuing in this way we get $ef(x)=0$ and so $R[[x;\alpha]]e \subseteq l_{R[[x;\alpha]]}(I)$. Proof of remaining part of this Theorem is similar to Theorem 2.2 Hence $R[[x;\alpha]]$ is quasi-Baer.

(2) $\Rightarrow$ (3) It is clear.

(3) $\Rightarrow$ (1) Similar to Theorem 2.2.

Corollary 4.2 ([Theorem 3.28]14) *Let R be an α-weakly rigid ring. If R is a quasi-Baer ring then $R[[x;\alpha]]$ is a quasi-Baer ring.*

Motivated by Pourtaherian and Rakhimov [15], we define quasi α-Armendariz ring of power series type as follows:

Definition 4.3 *Let R be a ring and α be an endomorphism of R. Then R is called a quasi α-Armendariz ring of power series type if for $p=\sum_{i=0}^{\infty} a_i x^i, q=\sum_{i=0}^{\infty} b_j x^j \in R[[x;\alpha]]$, $pR[x;\alpha]q=0$ implies $a_i R b_j = 0$ for all i and j.*

Theorem 4.4 *Let R be a quasi α-Armendariz of power series type. If R is α-quasi-Baer ring then $R[[x;\alpha]]$ is a quasi-Baer ring.*

Proof. Let R be a quasi α-Armendariz of power series type and α-quasi-Baer ring, and let I be any arbitrary ideal of $R[[x;\alpha]]$. Consider I_0 be the set of all the coefficients of elements of I. Observe that I_0 is an α-ideal of R since for $f(x)=\sum_{i=0}^{\infty} a_i x^i \in I$ and $g(x)=x \in R$, we have $g(x)f(x)=\sum_{i=0}^{\infty} \alpha(a_i)x^{i+1} \in I$ and so $\alpha(a_i) \in I_0$ for each i. Thus I_0 is an α-ideal of R, which gives $l_R(I_0)=Re$ for some right semicentral idempotent $e \in R$ i.e. $ea_n = 0$ for any $a_n \in I_0$. Now we show that $l_{R[[x;\alpha]]}(I)=R[[x;\alpha]]e$. Suppose $f(x)=\sum_{i=0}^{\infty} a_i x^i \in I$, so $ef(x)=\sum_{i=0}^{\infty}(ea_i)x^i = 0$ so $R[[x;\alpha]]e \subseteq l_{R[[x;\alpha]]}(I)$. Again suppose $g(x)=\sum_{j=0}^{\infty} b_j x^j \in l_{R[[x;\alpha]]}(I)$ which implies $g(x)rf(x)=0$, it follows that $b_j r a_i = 0$ since R is quasi α-Armendariz of power series type. Then $b_j \in l_R(a_i) = Re$ which gives $b_j = b_j e$. Thus $g = ge$ and therefore $l_{R[[x;\alpha]]}(I) \subseteq R[[x;\alpha]]e$. Hence the result follows.

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