An Application of the APOS/ACE Approach in Teaching the Irrational Numbers

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Abstract

The APOS/ACE instructional treatment for learning and teaching mathematics was developed during the 1990's by a team of mathematicians and mathematics educators led by Ed Dubinsky. In the present article we design in terms of the APOS/ACE treatment a general plan for teaching the real numbers at an elementary level (high school and college introductory mathematical courses). Our didactic approach is based on multiple representations of real numbers and on flexible transformations among them. Two classroom experiments performed during the last two academic years with students of my institution
(T. E. I. of Patras, Greece) are also reported illustrating the effectiveness of our teaching design in practice.

Introduction

Research focussed on the comprehension of irrational numbers shows that, apart from the earlier incomplete comprehension of rational numbers, they are also other obstacles (cognitive and epistemological) making it even more difficult (Herscovics 1989, Sierpinska 1994, Sirotic & Zazkis 2007a, etc).

Fischbein et al. (1995) assumed that possible obstacles for the comprehension of irrational numbers could be the intuitive difficulties that revealed themselves in the history of mathematics, i.e. the existence of *incommensurable magnitudes* and the fact that the *power of continuum* of the set ***R*** of real numbers is higher than the power of ***Q***, which, although being an everywhere dense set, can not cover all points of a given interval. Their basic conclusion resulting from their experiments with school students and pre-service teachers was that, since school mathematics is generally not concerned with the systematic teaching of the hierarchical structure of the various classes of numbers, most of high school students and many pre-service teachers were not able to define correctly the concepts of rational, irrational and real numbers, neither to identify various examples of numbers. They also found that, contrary to their initial assumption, the concept of irrational numbers does not encounter in general a particular intuitive difficulty in students' mind. Hence they assumed that such difficulties are not primitive ones and they express a relatively high level of mathematical education. However they suggest that for a better understanding of irrational numbers the teacher should turn students' attention on these difficulties rather, than ignore them.

Peled & Hershkovitz (1999) performing an experimental research observed that pre-service mathematics teachers being at their second and third year of studies, although they knew the definitions and basic characteristics of the irrational numbers, they failed in tasks that required a flexible use of their different representations.

Sirotic & Zazkis (2007b) focusing on the ability of prospective secondary teachers in representing irrational numbers as points on a number line observed confusion between irrational numbers and their decimal approximation and overwhelming reliance on the latter. They also used (Zazkis & Sirotic 2010) the distinction between *transparent* and *opaque* representations of concepts (Lesh et al. 1987) as a theoretical perspective in studying the ways in which different decimal representations of real numbers influenced their responses with respect to their possible irrationality.

Voskoglou & Kosyvas (2011, 2012) report on a study of high-school and of technologist students (prospective engineers and economists) understanding of real numbers. The study was based on written response to a properly designed questionnaire and on interviews taken from students. The superiority of the technologist students' correct answers with respect to those of high-school students was evident in most cases. This is a strong indication that the age and the width of mathematical knowledge play an important role for the better understanding of the real numbers. The results obtained suggest also that the ability to transfer in comfort among several representations of real numbers helps students in obtaining a better understanding of them.

The purpose of this paper is to apply the APOS/ACE framework for learning and teaching mathematics, developed by a team of mathematicians and mathematics educators led by Ed Dubinsky (Asiala et. al. 1996), in providing plausible explanations of the students' difficulties in understanding the irrational numbers and a general plan for teaching them at school. The paper is organized as follows: In the second section we present the general lines of the APOS/ACE approach and in the third section we describe the APOS analysis for the concept of infinity, which is strictly connected with the better understanding of the irrational numbers. In the fourth section we design a general plan for teaching the irrational numbers through a theoretical analysis of the concepts involved in terms of the mental constructions a learner might take in order to develop understanding of the concepts. In the fifth section we present a classroom experiment by applying the constructed general plan in terms of the ACE style in teaching the irrational numbers to an experimental group of students and by comparing the results of this group with those of another group of students taught the same subject in the traditional way (control group). Finally, the sixth and last section of the paper contains the conclusions of our study and relevant discussion.

Description of the APOS/ACE instructional treatment

Dubinsky had already spent twenty- five years doing research in functional analysis and teaching undergraduate mathematics before starting his new career on figuring out pedagogical strategies that help students to be more successful in learning mathematics. APOS is a theory based on Piaget's principle that an individual learns (e.g. mathematics) by applying certain mental mechanisms to build specific mental structures and uses these structures to deal with problems connected to the corresponding situations (Piaget 1970). Thus, according to APOS theory, an individual deals with a mathematical situation by using certain mental mechanisms to build cognitive structures that are applied to the situation. The main mechanisms are called *interiorization* and *encapsulation* and the related structures are *actions, processes, objects* and *schemas.* The last four words constitute the acronym APOS.

The theory postulates that a mathematical concept begins to be formed as one applies transformations on certain entities to obtain other entities. A transformation is first conceived as an action. For example, if an individual can think of a function only through an explicit expression and can do little more than substitute for the variable in the expression and manipulate it, he (she) is considered to have an action understanding on functions.

As an individual repeats and reflects an action it may be interiorized to a mental process. A process performs the same operation as the action, but wholly in the mind of the individual enabling her/him to imagine performing the transformation without having to execute each step explicitly. For example, an individual with a process understanding of a function thinks about it in terms of inputs, possibly unspecified, and transformations of those inputs to produce outputs.

If one becomes aware of a mental process as a totality and can construct transformations acting on this totality, then we say that the individual has encapsulated the process into a cognitive object. In case of functions encapsulation allows one to form sets of functions, to define operations on such sets, to equip them with a topology, etc. Although a process is transformed into an object by encapsulation, this is often neither easy not immediate. This happens because encapsulation entails a radical sift in the nature of one's conceptualization, since it signifies the ability to think of the same concept as a mathematical entity to which new, higher-level transformations can be applied. On the other hand, the mental process that led to a mental object through encapsulation remains still available and many mathematical situations require one to *de-encapsulate* an object back to the process that led to it. This cycle may be repeated one or more times (e.g. going back from a composite function to its component functions for the better understanding of the rule of derivation of a composite function, going back from the derivative to the initial function in order to understand the process of the integration of a function, etc).

A mathematical topic often involves many actions, processes and objects that need to be organized into a coherent framework that enables the individual to decide which mental processes to use in dealing with a mathematical situation. Such a framework is called a schema. In the case of functions it is the schema structure that is used to see a function in a given mathematical or real-world situation.

The implementation of APOS theory as a framework for the learning and teaching of mathematics involves three stages. Firstly a theoretical analysis, called a *genetic decomposition* (GD) of the concepts under study, is performed.

The GD comprises a description that includes actions, processes and objects and the order in which it may be best for learners to experience them. Then instructional sequences based on the GD are developed and implemented and finally data is collected and analysed in order to test and refine the GD and the pedagogical strategies employed (Dubinsky & McDonald 2001).

The main contribution obtained from an APOS analysis is an increased understanding of an important aspect of human thought. However, explanations offered by such analyses are limited to descriptions of the thinking that an individual may be capable of and not of what really happens in an individual's mind, since this is probably unknowable. Moreover, the fact that one possesses a certain mental structure does not mean that he/she will necessary apply it in a given situation. This depends on other factors regarding managerial strategies, prompts, emotional state, etc.

The APOS theory has important consequences for education. Simply put, it says that the teaching of mathematics should consist in helping students use the mental structures they already have to develop an understanding of as much mathematics as those available structures can handle. For students to move further, teaching should help them to build new, more powerful structures for handling more and more advanced mathematics. Dubinsky and his collaborators realized that for each mental construction that comes out of an APOS analysis, one can find a computer task of writing a program or code, such that, if a student engages in that task, he (she) is fairly likely to build the mental construction that leads to learning the mathematics. In other words, performing the task is an experience that leads to one or more mental constructions. As a consequence of the above finding, the pedagogical approach based on APOS analysis, known as the *ACE teaching cycle*, is a repeated cycle of three components: *(A) activities on the computer, (C) classroom discussion and (E) exercises done outside the class*.

In applying the ACE cycle the mathematical topic under consideration is divided to smaller subtopics and each iteration of the cycle corresponds to one of the above subtopics. The computer activities, which form the first step of the ACE approach, are deigned to foster the students' development of the appropriate mental structures. The students do all of their work in cooperative groups. In the classroom the teacher guides the students to reflect on the computer activities and their relation to the mathematical concepts being studied. They do this by performing mathematical skills without using the computer. They discuss their results and listen to explanations by fellow students, or the teacher, of the mathematical meanings of what they are working on. The homework exercises are fairly standard problems related to the topic being studied. Students reinforce the knowledge obtained in the computer activities and classroom discussions by applying it in solving these problems.

The implementation of the ACE cycle and its effectiveness in helping students make mental constructions and learn mathematics has been reported in several research studies of the Dubinsky's team. A summary of earlier work can be found in Weller et al. (2003). More recently this approach was applied in studying the pre-service teachers understanding of the relation between a fraction or integer and its decimal expansion (Weller, Arnon & Dubinsky 2009, 2011).

The APOS analysis for the concept of infinity

The concept of infinity is strictly connected with the better understanding of the irrational numbers. In fact, it is well known that each irrational number is the limit of the sequence of its successive finite decimal approximations. Further, the concept of infinity is involved in understanding that the power of continuum of the set of real numbers is higher than the power of the set of rational numbers and in several other cases. Therefore, one wanting to apply the APOS/ACE framework for teaching the irrational numbers, he (she) must study first the APOS analysis for the concept of infinity.

Aristotle's (384-322 B.C.) potential/actual dichotomy dominated conceptions of the infinity for centuries. He defined the *actual infinity* to be the infinite present of a moment in time and he considered it to be incomprehensible because the underlying process of such an actuality would require the whole of time. He argued that infinite could only be understood as being presented over the time, that is, as being a *potential infinity*. Despite past and current favour toward Aristotle's views, there were some dissenters, such as the rationalists. Bolzano (1741-1848) played a major role in advancing the notion of an *infinite totality* when he rejected Aristotle's assertion that a collection does not exist as a completed whole unless one forms an image of every item, or reflects on every step of the process that generates it. In his view we can use our minds to conceive of an infinite collection by describing its elements without having to think of each element individually. His main argument supporting the above view is that, if we consider Aristotle's assertion to be true, then we must deny the existence of *large finite numbers*, like the grains of sand in a desert, etc. For a detailed account of historical and philosophical issues of infinity see Moore (1999).

Dubinsky et al. (2005 a, b) expanded the APOS theory to explain how people may think about the concept of infinity and to analyze many of the difficulties appearing in understanding this concept. Their analysis suggests that potential infinity is the conception of the infinite as a process, while actual infinity is the mental object obtained through the encapsulation of that process to an object. Hence the existence of the one does not negate the other, nor is a misconception with respect to the other. Instead they represent two different cognitive conceptions that are related to the mental mechanism of encapsulation. These conceptions and their relationship become part of the individual's infinity schema. In this sense we might argue that Bolzano's thinking is evidence in support of APOS analysis.

In the case of an infinite process the mental object transcends the process (e.g. limit of a sequence) in the sense that it is not associated with nor is produced by any step of the process (*transcendental object*). This is the cognitive difference between large finite numbers, where the last number enumerated indicates its completion (*final object*) and the infinite and explains why the former can be easier understood. The delay often seen in the development of encapsulation may explain why it took centuries for the actual infinity to be widely accepted, and why many still find the dichotomy between potential and actual infinity to be perplexing.

In concluding, we may say that through encapsulation the infinite becomes *cognitively attainable*. On the other hand *cognitively unattainable* is the instance of the infinite in the form of a process that has not been encapsulated. This may happen because the process has not yet seen as a totality, either because it <u>cannot</u> be seen as a totality, or because no encapsulation has taken place. Thus, for the cognitive point of view it must be understood that for the case of infinity the ability to see something as a totality and the mechanism of encapsulation may not always been available (as in the set of all sets and other unmanageable sets). This resolves various paradoxes that appear in the concept of infinity.

A general plan for teaching the irrational numbers based on the APOS/ACE approach

In our introduction we referred to the difficulties appearing to learners in comprehending the irrational numbers. In this section and in designing a GD for irrational numbers we shall attempt to give a theoretical explanation about them in terms of the APOS theory. Further, we shall propose possible ways to overcome these difficulties.

An essential pre-assumption for the comprehension of irrational numbers is that students have already consolidated their knowledge about rational numbers and, if this has not been achieved, as it usually happens, many problems are created. It has been observed that pupils, but also university students at all levels, are not able to define correctly the concepts of rational and irrational numbers, neither are in position to distinguish between integers and these numbers (Hart 1988, Fischbein et al. 1995). It seems that the concept of rational numbers in general remains isolated from the wider class of real numbers (Moseley 2005, Toepliz 2007). But why all these happen? Let us start from the notion of a fraction. If someone can think of a fraction, for example $\frac{2}{3}$, only by dividing a specific object (e. g a chocolate) in 3 equal pieces and pick 2 of them, then he/she has an action conception of fractions. Most students as a result of the normal human activity are able, after repeating such an action and reflecting on it, to build a process conception of fractions that allows them to imagine dividing an unspecified object in 3 parts an taking 2 of them. However the encapsulation of this process to a mental object does not seem to be so easy. This explains why many students consider the multiple ways of writing down a fraction as being different fractions, or why they consider fractions and decimals as being different kinds of numbers. The latter explains also why they are not in position to distinguish between integers and decimals.

On the other hand the frequent identification by students of real numbers with their given finite approximations (e.g. identification of π with *3,14* or with

$\frac{22}{7}$, of $\frac{144}{233}$ with *0,180257*, when performing the division *144:233* with a calculator, etc) means that these students have an action understanding of the notion of real number that has not been interiorized yet to a mental process, i.e. to potential infinity.

We can see reflections on the development of the concept of actual infinity in students of today. Nunez (1993) reporting on a study of the construction of infinite processes by children aged 9-14 notes that none of his subjects shoed any signs of thinking about an actual infinity: all their comments were in terms of potential infinity. He suggests that the reason is that the concept of actual infinity does not arise before the age of 15. His view is supported by results of Haucart and Rouche (1987) who found that some students aged 12-18 did seem to have a concept of actual infinity. These authors discuss the relation between potential and actual infinity in terms of the movement from an infinite process to its limit. All the above lead to the conclusion that, since the notion of an infinite decimal is obtained as the transcendental object of an infinite process (limit of a sequence of finite decimals) the encapsulation of this process is not easy. This gives an explanation of students' difficulties in defining correctly the rational and irrational numbers.

However, the encapsulation of the concept of real numbers is getting more perplexed due to the fact that many rational numbers, like $\frac{144}{233} = 0{,}61802575107\ldots$, which possesses a period of 232 digits, and most of irrational numbers have *opaque*, decimal representations (Lesh, Behr, and Post 1987, Zazkis and Sirotic 2010). This makes the recognition of their possible rationality or not to be impossible when only their decimal representations are given. But, for students *it is difficult in general to understand a number, if they don't know an explicit way of writing it down*. In fact, it seems that people tend to adapt their formal knowledge to accommodate their beliefs (i.e. the conclusions of their intuitive knowledge), perhaps as a natural tendency towards consistency. Therefore, when their beliefs are not clear and/or accurate, as it happens with opaque representations of real numbers, it is very possible to lead to mistakes and/or inconsistencies.

Concerning the difficulties in accepting the existence of incommensurable magnitudes and in understanding that the power of continuum of $\boldsymbol{R}$ is higher than the power of $\boldsymbol{Q}$ they could be considered as typical cases of cognitively unattainable infinity, where the infinite process involved is not yet seen as a completed totality and therefore it has not been encapsulated.

The problems are increasing when dealing with the existence of transcendental numbers. In fact, students already know that the periodic decimals can be alternatively written as fractions. Later they learn that the square roots of non quadratic fractions are irrational numbers and that the same happens with the roots of rational numbers of any order whose value is not a finite decimal. Therefore it is difficult afterwards to accept the existence of irrational numbers that cannot be traced algebraically. In other words this is another case of a mental process that cannot be seen easily by students as a completed totality and therefore to be encapsulated to a mental object (of irrational numbers). Finally, the students' difficulty in dealing with comfort with the multiple representations of real numbers is a consequence of the inadequate organization by them of a powerful schema for real numbers.

Reflecting on the above explanations of students' difficulties in understanding the real numbers we designed a general plan for teaching the irrational numbers consisting of three iterations of the ACE cycle. Each cycle consisted of two class days, one for computer activities and one for classroom discussions. Homework exercises were assigned and collected. Notice that, since the proper understanding of the rational numbers is an essential pre-assumption for the comprehension of the irrational numbers, our design involved frequent repetitions of the corresponding situations for rational numbers. Some of these repetitions were adapted from Weller et al. (2009).

In an action level the concept of an infinite decimal (rational or irrational number) is understood by considering its finite decimal approximations. The target of the *first iteration* of the ACE cycle was to facilitate the interiorization of this action to a process. The students completed in the computers' laboratory activities with a preloaded decimal expansion package. They developed general descriptions of what was stored and answered various questions about an infinite digit string such as: What is a repeating decimal? Which of the strings are repeating decimals? What are the digits in the first 20 places after the decimal point and what would appear in the 1005^{th} place? Further, students were asked from to calculate the successive finite decimal approximations of several square roots with gradually increasing accuracy.

In the classroom discussion the students reported their group responses and the class negotiated agreements. A notational system for infinite decimals was devised. For example, since $1<\sqrt{2}<2$, $1{,}4<\sqrt{2}<1{,}5$, $1{,}41<\sqrt{2}<1{,}42$, $1{,}414<\sqrt{2}<1{,}415$, $1{,}4142<\sqrt{2}<1{,}4143$, $1{,}41421<\sqrt{2}<1{,}41422$, etc, $\sqrt{2}$ can be written as $\sqrt{2} = 1{,}4142.....$. The dots at the end indicate that the sequence of the decimal digits is continued to infinity. Therefore, by accepting this symbolic representation of an infinite decimal we can not see written all its decimal digits. We can only see the digits of its given decimal approximation each time. The instructor recalled at this point that a repeating decimal (rational number) can be written in the form $a,b\overline{c}$. Here *a, b, c* are natural numbers, where *a* denotes the integer part of the rational number, *b* is its decimal portion that possibly appears before the repeating cycle (in case of mixed periodic numbers) and *c* is the repeating cycle (period) of the number. A finite decimal can be written as a repeating decimal with period *0* or *9*; e.g. $2{,}5 = 2{,}5\overline{0} = 2{,}4\overline{9}$ The exercises included problems where certain information about an infinite digit string was provided that was sufficient to specify the string.

The target of the *second iteration* of the ACE cycle was to facilitate the encapsulation of the concept of a real number to a mental object. During the computer activities students were asked to work out examples with *transparent* and *opaque* decimal representations of real numbers like the following: The rational numbers $\frac{3}{5}=0{,}6$, $\frac{1}{3}=0{,}33...$, $\frac{281849}{99900}=2{,}82113113113...$, have transparent decimal representations, since we can foresee their decimal digits in all places; but the same is not happening with $\frac{1}{1861}$= 0,0005373…, which, possessing a period of 1860 digits, has an opaque decimal representation. Notice that decimal representations of certain irrational numbers, despites to their complex structure in general, are also transparent. For example, this happens with the numbers *2,0013131131111311113111113…* where *1*, following *13*, is repeated one more time at each time, and *0,2822882228882222888882…*where *2* and *8*, following *28*, are repeated one more time at each time. Taking this opportunity the instructor clarified to the class that an infinite decimal is an incommensurable (non periodic) decimal not because its decimal digits are not repeated in a concrete process (this in fact can happen according to the above two examples), but because it has not a period, i.e. its decimal digits are not repeated in the same concrete series. Some standard cases of decimal expansions of transcendental numbers like π and *e* were also added to the above examples. Students were also asked to convert fractions and roots of second or higher order to decimals and vice versa. Further, the computer activities included arithmetic operations among irrational and rational numbers by using their finite decimal approximations.

In the classroom the students performed the same mathematical skills without using the computers. In this way they realized that in converting a fraction to a decimal, if the quotient obtained is an infinite decimal having a long period, a long and laborious division is reached in general, which is not possible to be determined soon. At this point the instructor emphasized that given a fraction $\frac{\mu}{\nu}$, $\mu,\nu \in Z$, $\nu \neq 0$, the quotient of the division $\mu : \nu$ is always a periodic decimal. The probability to be a finite decimal is small enough, since a fraction, whose denominator is not a product of powers of *2* and/or *5*, cannot be written as a finite decimal. In case of an infinite decimal, since the remainder of the division $\mu : \nu$ is smaller that ν, performing the division and after a finite number of steps (at most ν*-1*) the same remainder will reappear at some step. This means that the resulting decimal is a periodic one, having a period of at most ν*-1* digits. Conversely, a standard method for converting periodic numbers to fractions (although they could be used other methods as well) is by subtracting both members of proper equations containing multiples of a power of *10* of the given number. For example, given *x=2,75323232…*, we write *10000x=27532,3232…* and *100x=275,3232….*, wherefrom we get that *9900x=27532-275*, or $x=\frac{27257}{9900}$. Reflecting on the above examples the students reached to the conclusion that periodic decimals and fractions are the same numbers written in a different way. Students' contact at school with the definition of irrational numbers as *incommensurable decimals* is usually rather slim, while emphasis is given in defining them as *non rational numbers* (i.e.

they cannot be written as fractions $\frac{\mu}{\nu}$, with μ, ν integers and $\nu \neq 0$). However, students must clearly understand the equivalence between the above two definitions: Since rational numbers and periodic decimals are the same numbers written in a different way, the same thing holds for non rational numbers and incommensurable decimals. Thus, the set of real numbers ***R*** can be defined as the set of all commensurable and incommensurable decimals and their opposites. In closing the classroom discussion the instructor presented empirically the concept of a sequence of finite decimals and of its limit (i.e. what it means to "tend" to a number) and explained it to students by using the appropriate examples, like this with $\sqrt{2}$ mentioned above. In no case it becomes necessary for the teacher to give the analytic definition of the limit of a sequence. The above empiric approach is enough in helping students to encapsulate the concept of a real number to a mental object. The homework exercises were standard problems related to the topics mentioned above aiming to consolidate the students' knowledge and understanding of these topics.

The target of the *third* iteration of the ACE cycle was to help students to enlist the real numbers in general and the irrational numbers in particular in their cognitive schema related to the already known basic sets of numbers (natural numbers, integers and rational numbers). A prerequisite for this is that they must be able to transfer in comfort among the several representations of real numbers. Therefore, the computer activities in this cycle involved among the others examples of constructions of line segments with incommensurable lengths; either classical geometrical constructions by using the Pythagorean theorem, like $\sqrt{2}$, $\sqrt{3}$ etc, or cases where the construction of the graph of a function is necessary, like $\sqrt[3]{2}$ with the function $f(x)=\sqrt[3]{x}$ (or $f(x)=\sqrt[3]{x}-2$) etc. They involved also examples of writing real numbers in the form of a series $x=\sum_{n=0}^{\infty}\frac{\kappa_n}{10^n}$, where κ_0 is an integer and κ_1, κ_2,…., κ_n,…. are natural numbers less than 10*. Finally, the computer activities involved also examples of interpolation of rational and irrational numbers between two given integers, or between two rational (irrational) numbers aiming to promote the later discussion in classroom about the density of the sets of rational and real numbers.

In the classroom discussion the instructor recalled first that in defining the set **Q** of rational numbers as the set of all fractions and in order to count each fraction only once, we considered only the fractions of the form $\pm\frac{\mu}{\nu}$, where μ

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*If in the above series we have $\kappa_1=\kappa_2=\ldots..=\kappa_\nu=------=9$, it is easy to check that $x=\kappa_0+1$. Therefore, if we denote by $[x]$ the integral part of x, we have that $[x] = \kappa_0$ and at the same time that $[x] = \kappa_0+1$, which is absurd! Therefore there is a debate in the literature whether or not decimal expansions of the form $\kappa_0,\bar{9}$ are representing real numbers; e.g. see Voskoglou 2012. Fortunately the results obtained when using these representations are conventionally correct because the corresponding operations could be performed in an analogous way among the sequences of the partial sums of the corresponding series. This allows us at school level to pass through this sensitive matter without touching it at all.

and v are non negative integers ($v \neq 0$), with greatest common divisor equal to *1*. In an analogous way, since for all integers κ and α with $1 \leq \alpha \leq 9$ we have $\kappa,\alpha = \kappa,(\alpha-1)\overline{9}$ and $\kappa,\overline{9} = \kappa+1$, in defining ***R*** as the set of all decimals and in order to count each real number only once , we must exclude all infinite decimal expansions of the form $\kappa,\kappa_1\kappa_2$…….. , in which there exists a natural number v such that $\kappa_\mu=9$ for all $\mu \geq v$. The instructor presented also to students some details about the *transcendental* numbers. This new kind of numbers usually activates students' imagination and increases their interest by creating a pedagogical atmosphere of mystery and surprise. It can be shown that the set of algebraic numbers is a denumerable set, while Cantor proved that the set of transcendental numbers has the power of continuum. This practically means that transcendental are much more than algebraic numbers, but the information that we have about them is very small related to their multitude. That is why one can characterize them as a "black hole" (in analogy with the astronomical meaning of term) in the "universe" of real numbers (Voskoglou 2011).

Activities on geometric constructions of irrational numbers were also organized in classroom combining history of mathematics with Euclidean Geometry. Within the culture of ancient Greek mathematics the geometric figure was the basis for unfolding mathematical thought, since it helped in obtaining conjectures, fertile mathematical ideas and justifications (proofs). In fact, convincing arguments are built by drawing auxiliary lines, optical reformations and new modified figures, and therefore mathematical thinking becomes more completed in this way. Therefore the geometric representations of real numbers enrich their teaching, connecting it historically with the discovery of incommensurable magnitudes and the relevant theory of Eudoxus. Following these historical steps of the human thought is probably the best way in helping students to accept the existence of incommensurable magnitudes. Another crucial matter for the instructor is to find the proper way to explain to students the continuum of ***R*** with respect to the density of ***Q***. In other words to persuade them that in a given interval of numbers it is possible to have an infinite number of elements of a certain type (rational numbers) and at the same time to be able to add an infinity of elements of another type (irrational numbers), when this is not compatible with the usual logic and intuition. It seems that the use of the geometric representations of real numbers is the best way to deal with this problem (an interval of points on the real axis cannot be "filled" with rational points only). The difficulty in this case is that most of the irrational numbers, like $\sqrt[3]{2}$, π, e, etc, correspond to lengths of line segments that cannot be constructed geometrically. Therefore, we correspond to all these numbers points of the real axis in an approximate way by using their finite decimal approximations and our fantasy[1].

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[1] Mathematically speaking the above correspondence is based on the *principle of the nested intervals* connected to the method of *Dedekind cuts* for defining the real numbers (e.g. see Voskoglou & Kosyvas 2011; section 2), an approach not compatible with an elementary presentation of real numbers to students. Nevertheless, in my country (Greece) during the period of bloom of "modern mathematics" in school education (1970-1990) the presentation of ***R*** as an ordered field was attempted at the upper high school level (Lyceum), where the principle of nested

In concluding, our general didactic approach included: A fertile utilization of already existing informal knowledge and beliefs about numbers, active learning through rediscovery of concepts and conclusions, construction of knowledge by students individually or as a team in the computers' laboratory and in classroom. Construction of knowledge followed in general student's optical corner, while teacher's role was limited to the discussion in the whole class of wrong arguments and misinterpretations observed. The teaching process was based on multiple representations of real numbers (rational numbers written as fractions and periodic decimals, irrational numbers considered as non rational ones and as incommensurable decimals which are limits of sequences of rational numbers, geometric representations, etc) and on flexible transformations among them. It was hoped that this approach could help students in building a powerful schema for real numbers.

A classroom experiment

In developing and applying in practice our ACE design for teaching the irrational numbers we performed during the academic year 2011-12 a classroom experiment with two groups of students of the Graduate Technological Educational Institute (T. E. I.) of Patras, Greece being at their first term of studies. The subjects of the experimental group were 90 students of the School of Technological Applications (prospective engineers) attending the course "Higher Mathematics I" [2]. The students of this group were taught the irrational numbers in the computers' laboratory and in the classroom according to our ACE design presented above. The subjects of the control group were 100 students of the School of Management and Economics attending a similar mathematical course (the instructor was the same person). In this group the lectures were performed in the classical way on the board, followed by a number of exercises and examples. The students participated in solving these exercises.

On the first day in class the students of both groups completed individually a five-item pre-instructional written questionnaire (see Appendix I). The instrument served to establish the similarity of the two groups and to guide the development of the teaching process. The students of the two groups responded similarly to the questionnaire items. At the end of the instructional unit students of both groups completed a new ten-item post-instructional written questionnaire (see Appendix II). Students were instructed to work on the

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intervals, presented as a fundamental axiom, was used in defining real numbers. It was proved impossible however to persuade students in this way that each sequence of nested intervals leads to the construction of a unique real number. This strict axiomatic approach, isolated from Dedekind's cuts (at any case such a connection could not be presented at school level) didn't offer any more to the better understanding of the real numbers, than their decimal representations do. On the contrary, it caused a great confusion to students.

[2] The course involves an introductory chapter repeating and extending the students' knowledge from secondary education about the basic sets of numbers, Complex Numbers, Differential and Integral Calculus in one variable, Elementary Differential Equations and Linear Algebra.

questionnaire individually and to answer each question thoroughly. The instrument counted as a progress grade added to the course's final exam results. In assessing the general performance of the two groups we applied the widely known GPA method [3]. Now the performance of the first (experimental group) was found to be significantly better (GPA_1 □ 3,19 and GPA_2 = 2,64). An analogous experiment was repeated during the current academic year 2012-13. This time, although the students of the control group responded slightly better to the pre-instructional questionnaire, the superiority of the experimental group was evident again concerning the post-instructional questionnaire GPA_1 □ 2,95 and GPA_2 □ 2,61) .

In concluding, the results of our experiments give a strong indication that the use of the ACE cycle as a teaching method enhances the students' understanding of real numbers in general and of irrational numbers in particular.

Conclusions and Discussion

Based on those reported in the present paper the following conclusions can be drawn:

- The understanding of real numbers by students strikes against inherent difficulties, connected to the incomplete earlier understanding of rational numbers and the nature of irrational numbers.
- The APOS theory and its analysis for the concept of infinity provide plausible explanations for the above difficulties. Reflecting on these explanations we designed in terms of the APOS/ACE approach a general plan for teaching the real numbers at an elementary level (high school and college introductory mathematical courses). Our didactic approach was based on multiple representations of real numbers and on flexible transformations among them.
- Our classroom experiments, performed the last two academic years at the T.E.I of Patras, Greece, give a strong indication that the application of the APOS/ACE approach for teaching the real numbers in general and the irrational numbers in particular can help effectively students in building a powerful cognitive schema for the basic sets of numbers.

Notice that, in contrast to the ancient Greek mathematics, numerical thought is the most frequently used at school today. This can be logically explained, since numerical excels geometrical culture in our contemporary world and therefore it plays the main role in representations that students build at school. Nevertheless, we have the feeling that in general the excessive use of

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[3] The **G**reat **P**oint **A**verage (GPA) is a weighted average of the students' performance. For this, each student's paper is marked with A (90-100%), B (80-90%), C(80-70%), D (60-70%), or F (< 60%). Then, if n is the total number of students and n_A, n_B, n_C, n_D, n_F denote the numbers of students getting the marks A, B, C, D, F respectively, GPA = $\frac{0.n_F + 1.n_D + 2.n_C + 3n_B + 4.n_A}{n}$.

Obviously we always have $0 \leq \text{GPA} \leq 4$.

numerical arguments wounds the geometrical intuition. In fact, we believe that a rich experience of students with geometric forms, before being introduced to numerical arguments and analytical proofs, is not only useful, but it is actually indispensable (see also Arcavi et al. 1987). However, we ought to clarify that all those discussed in this article must be simply considered as a series of well organized ideas aiming to help the instructor towards the difficult indeed subject of teaching the real numbers at an elementary level. In no case they should be considered as an effort to impose a model of teaching the real numbers. In fact, our general belief is that the teacher should be able to make a small "local research", readapting methods and plans of the teaching process according to the teaching environment and the special conditions of each class (Voskoglou 2009, section 3). Obviously they remain some open questions for future research, the most important being probably how students could understand better the approximate correspondence of incommensurable magnitudes, which cannot be geometrically constructed, to points of the real axis. Among our future research plans is also the performance of more classroom experiments on the subject with different groups of students (high school students as well) in order to obtain statistically stronger results and conclusions.

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Appendix I: Pre-instructional questionnaire

1. Which of the following numbers are natural, integers, rational, irrational and real numbers?

$$-2\,,\quad -\frac{5}{3}\,,\quad 0\,,\quad 9{,}08\,,\quad 5\,,\quad 7{,}333...\,,\quad \pi = 3{,}14159...\,,\quad \sqrt{3}\,,\quad -\sqrt{4}\,,$$

$$\frac{22}{11},\quad 5\sqrt{3},\quad -\frac{\sqrt{5}}{\sqrt{20}}\,,\quad \left(\sqrt{3}+2\right)\left(\sqrt{3}-2\right),\quad -\frac{\sqrt{5}}{2},\quad \sqrt{7}-2,$$

$$\sqrt{\left(\frac{5}{3}\right)^{2}}$$

2. Are the following inequalities correct, or wrong? Justify your answers.

$\frac{2}{3} < \frac{14}{21}$, $\frac{2002}{1001} > 2$

3. Convert the fraction $\frac{7}{3}$ to a decimal number. What kind of decimal number is this and why we call it so? .

4. Find the integers and the decimals with one decimal digit between which lies $\sqrt{2}$. Justify your answers.

5. Find two rational and two irrational numbers between 10 and 20. How many irrational numbers are there between these two integers?

Appendix II: Post-instructional questionnaire

1. Which is the exact quotient of the division *5:7*?

2. Are *2,8254131131131…* and *2,00131311311131111…* periodic decimal numbers? In positive case, find the period and convert the corresponding number to a fraction.

3. Find the square roots of *9, 100* and *169* and describe your method of calculation.

4. Characterize the following expressions by C if they are correct and by W if they are wrong: $\sqrt{2} = 1{,}41$, $\sqrt{2} = 1{,}414444\ldots$, $\sqrt{2} \approx 1{,}41$, there is no exact price for $\sqrt{2}$.

5. Find two rational and two irrational numbers between $\sqrt{10}$ and $\sqrt{20}$. How many rational numbers are there between these two square roots?

6. Are there any rational numbers between $\frac{1}{11}$ and $\frac{1}{10}$? In positive case, write down one of them. How many rational numbers are between the above two fractions?

7. Are there any rational numbers between *10,21* and *10,22*? In positive case, write down one of them. How many rational numbers are in total between the above two decimals?

8. Characterize the following expressions as correct or wrong. In case of wrong ones write the corresponding correct answer.

$\sqrt{3+5} = \sqrt{3} + \sqrt{5}$, $\sqrt{3 \cdot 7} = \sqrt{3} \cdot \sqrt{7}$, $\sqrt{\frac{2}{9}} = \frac{\sqrt{2}}{3}$, the unique solution of the equation $x^2 = 3$ *is* $x = \sqrt{3}$, $\sqrt{\left(1-\sqrt{17}\right)^2} = 1 - \sqrt{17}$

9. Construct, by making use of ruler and compass only, the line segments of length $\sqrt{5}$ and find the points of the real axis corresponding to the real numbers $\sqrt{5}$ and $-\sqrt{5}$. Consider a length of your choice as the unit of lengths.

10. Is it possible for the sum of two irrational numbers to be a rational number? In positive case give an example.