The Role of Proof in the Curriculum

David C. Bramlett, Ph.D. †
Carl T. Drake, Ph.D. ‡

Abstract

This paper will examine the role of proof in mathematics education. The debate on whether to teach mathematical proofs in kindergarten through grade twelve has been one of many opposing opinions and views. Over the last hundred years, the debate's primary focus in mathematics teaching was on teaching with understanding versus the teaching of facts and procedures, and the role that proofs have in the learning of mathematics.

Introduction

Whether to include mathematical proofs in kindergarten through grade 12 curriculums has been an ongoing debate for many years. Proofs in the geometry class have been the norm since the mathematics' reform of the 1890's. It was at this time that geometry became the place for students to learn the art of demonstration. Under the guidance of the Committee of Ten, proofs would become the centerpiece of the high school geometry course. Writing proofs in a two column format of statements and reasons was developed by teachers in response to the reform. This would establish the tradition in the American education system of geometry being the place where students would learn the skills of doing proofs in mathematics (Herbst, 2002).

The "New Math" curriculum of the late 1950's and early 1960's placed a greater emphasis on the concept of proofs. The key objectives of the New Math curriculum were to raise the standards and to move away from the "dumbing" down of mathematics that marked the beginning of the twentieth century. The mathematics of 1950 America was marked with inconsistency of terminology and language. The New Math curriculum wanted to organize the terminology and language to be used in the mathematics classroom. They wanted the students to be able to think and develop their own ways of solving a problem rather than just memorize and regurgitate an answer (Kilpatrick, 1997). The level of abstractness and the formal nature of some of the materials intimidated many of the teachers who received little or no training. This resulted in the public criticisms of the New Math curriculum (Klein, 2003). After the failure of the new math reform movement, proofs were once again deemphasized and students once again had little experience with mathematical proofs outside of a high school geometry class.

The late 1980's saw the beginnings of another mathematics reform movement. A key publication for this current reform was the 1989 Curriculum and Evaluation Standards published by the National Council of Teachers of Mathematics (NCTM). Between 1990 and 2003 there was an increase in papers on the teaching and learning of proofs in a mathematics class. In 1997 Nicolas

Balacheff began maintaining the web site, the International Newsletter on the Teaching of the Teaching and Learning of Mathematical Proofs which has been visited over 5000 times (Hanna, 2000).

Many mathematicians on both sides of the current reform movement can at least agree on the need for mathematical proofs in kindergarten through grade 12 mathematics education (Knuth, 2000). Problems arise among mathematics educators when it comes to the type and the rigor of the proofs that students should study. The National Council of Teachers of Mathematics (NCTM) has included proofs has a standard in their 1998 and 2000 Principles and Standards for School Mathematics publications. This has placed a new emphasis on the inclusion of the concepts of proofs in kindergarten through grade 12 education.

The Common Core State Standards Initiative (CCSS) released in 2010 is the newest of reforms in education in an attempt to establish national educational standards to make certain that students have a consistent educational experience no matter what state that they may live within the United States and its territories. The Common Core Standards Initiative has currently been adopted by 45 states and 3 territories and places a greater emphasis on education that is more relevant to the real world and reflecting the knowledge and skills that students will need to pursue a college education and to have a successful career in the ever-changing global workplace.

Mathematical proofs are an important aspect of mathematics. It is important that mathematics educators understand the role of proofs in teaching so that students can develop a better understanding and appreciation for mathematical proofs (Hanna, 2000).

The Expectation of Proofs

Deutsch (2000) defined proof as the following; "Proof is the method mathematicians use to communicate the truth or validity of a piece of knowledge. While the specific form proof takes may have undergone transformations over the centuries – generally changing as the kind of mathematics being done has changed – the notion of communicating an important result through an argument meant to convince the reader of the validity of the result is still at the heart of doing proof and in fact, doing mathematics. . And the notion of proof transcends mathematics – logical argument is at the heart of philosophy, the law, rhetoric and debate. The idea of proof certainly influences the scientific method and is part of the formal presentation scientists use to forward a new idea or result. The value of teaching students the fundamentals of mathematical proof can not be denied."

Hanna (2000) compiled the following as a list of expectations that should result when one proves in the mathematics classroom; (1) Verification and concern of the truth of a statement; (2) Explanation which will provide insight to why something is true; (3) Systematization which is the organization of a variety of results into a deductive system of axioms, major concepts, and theorems; (4) Discovery which is the finding or invention of new results; (5)

Communication which is the transmission of mathematical knowledge; (6) Construction of empirical theory; (7) Exploration into the meaning or the consequences of an assumption; and (8) Incorporation of well known facts into a new framework and therefore viewing it from a new perspective.

Both the 1998 and 2000 versions of NCTM's Principles and Standards for School Mathematics have included proofs as a standard for kindergarten through grade 12 mathematics education. NCTM viewed a mathematical proof as a way of expressing a variety of types of reasoning and a means of justifying concepts in mathematics (NCTM, 2000).

CCSS (2010) focuses on students being able to explain their work. Starting in elementary school, students will have to be able to explain their answer and can no longer just give an answer. They then will take this skill to justify their work through formal proofs as they enter high school.

Reasoning

In the NCTM publication Curriculum and Evaluation Standards (1989), reasoning was cited as one of the major goals of mathematics education from kindergarten through grade 12. NCTM felt that as students developed their reasoning skills that this would directly prepare them for proofs in mathematics. Kindergarten through grade 4 students would be involved in informal reasoning. Reasoning at this stage of development would involve informal thinking, conjecturing, and validity that would help the children gain a better understanding of mathematics. In grades five through eight the students would then begin working on formal reasoning and abstractions. In grades nine through twelve the students then apply their reasoning skills to construct proofs for mathematical assertions which would include indirect proofs and proofs by mathematical inductions (NCTM, 1989).

Reasoning has been an essential objective of mathematics for quite a long time. In the 1923 report entitled Reorganization of Mathematics in Secondary School Report, from the Mathematical Association of America, one can find reasoning mentioned. The Progressive Education Report and NCTM both in 1940 emphasized the importance of mathematical reasoning. Also in 1958, the College Entrance Examination Board report stressed the importance of mathematical reasoning (Franklin, 1996).

When then 1989 Standards were released, the role of proofs in the curriculum had all but disappeared. The NCTM standards did not try to completely attack this situation; instead they placed greater emphasis on testing conjectures, formulating counter examples, the construction and examination of valid arguments, as well as the abilities to use these skills to in non-routine problem solving (Hanna, 2000).

Because reasoning was emphasized instead of proof, the 1989 standards came under criticism from university professors such as Dr. H. Wu of the University of Berkley (Knuth, 1996). Wu (1996) felt that the role of proofs in the mathematics curriculum had essentially been reduced into a meaningless ritual. It was also felt that the 1989 Standards failed to truly utilize the power of proof as a teaching tool (Hanna, 2000). As an answer to the criticisms, NCTM would revise and upgrade Principles and Standards to include Reasoning and Proof as a major goal across all grade levels.

Reasoning and Proof

The 2000 NCTM Principles and Standards has remedied this problem by recommending that that reasoning and proof be an essential part of the mathematics curriculum across all grade levels from prekindergarten through grade twelve. Under the Reasoning and Proof section of the NCTM Principles and Standards, it is stated that all students should be able to: (1) recognize reasoning and proof as fundamental aspects of mathematics; (2) make and investigate mathematical conjectures; (3) develop and evaluate mathematical arguments and proofs; and (4) select and use various types of reasoning and methods of proofs (NCTM, 2000).

Ross (1998) states "that the foundation of mathematics is reasoning. While science verifies through observation, mathematics verifies through logical reasoning." Because of this foundation, the essence of mathematics can be found in proofs. Students should be taught the difference between illustrations, conjectures, and proofs. Students should be able to construct valid arguments and proofs and also be able to criticize arguments. If students do not learn to reason and prove, mathematics will then become simply a process of following procedures and mimicking examples without any concern of why it is valid or true (Ross, 1998).

The American Mathematical Society Association Resource Group (AMSARG) on NCTM Standards in 2000 stated that "mathematical reasoning, deduction, and formal proof are part of the nature of mathematics and should be part of the mathematics curriculum." They defined nature as an essential element which must be a part of the subject matter.

CCSS (2010) has as two of its primary focuses the need for students to reason abstractly and quantitatively, construct viable arguments, and critique the reasoning of others throughout grades K-12 with formal proofs becoming an essential part of the high school curriculum. It is also expected that all students in grades K-12 develop the ability to explain and justify their mathematical solutions.

Students should be given the opportunity to explore why something is true and to debate faulty reasoning and to explore their ideas on the topic. Mathematical reasoning should start in elementary school and advance along with the students. Mathematical reasoning should be a part of all mathematics classes and not just a geometry class. Students should be able to distinguish between a mathematical proof and simply a collection of examples. They should also be able to distinguish between mathematical statement which can be proved and a useful mathematical model. The primary aim for proofs in mathematics is to achieve mathematical understanding (AMSARG, 2000).

Proof is a significant part of the reasoning process and simple proofs should be included with the intermediate grades. When students develop their ability to reason, they are climbing to towards the highest level of Bloom's taxonomy of learning. At this point, there is a need to prove that what they have reasoned is always true. Proofs can be in various forms such as an argument using prose, a logic argument, an indirect proof which assumes that a statement is false in which one reaches a contradiction, or a proof using mathematical induction (Franklin, 1996).

Proof and Understanding

Carpenter and Lehrer (1999) listed the following mental activities that will result in understanding. For students to understand they must be able to construct relationships with there acquired knowledge. They must be able to extend and apply their mathematical and scientific knowledge. Students need to reflect about their mathematical and scientific experiences. Students should be able to articulate what one knows. Mathematical and scientific knowledge must be made into your own.

Reflection and communication are essential to building understanding. Reflection is the process of consciously thinking about your experiences. When you reflect, you are thinking about things from various points of view. You are in a sense stepping back to see the big picture to recognize and build relations between ideas, facts, and procedures. The result is that you will reanalyze old relations and create new relations resulting in new understanding (Hiebert, 1997).

When one begins to communicate ideas and information, this social interaction with others allows for the opportunity to share ideas, facts, and procedures with others. This will allow for you to receive suggestions and to have your ideas' challenged so that you must provide further explanation and clarification resulting in new understanding. As you reflect and communicate you are building understanding which in turn provides connections for the various ideas, facts, and procedures that you have obtained (Hiebert, 1997).

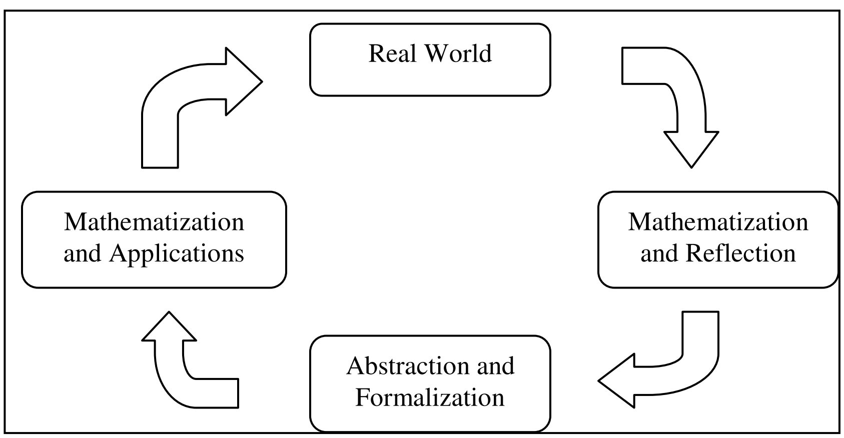
Proof and Students

In dealing with mathematical proofs, the skills which are necessary for building understanding in mathematics are all essential for proving in mathematics. This is why some researchers feel that proof is an essential part of understanding mathematics and its concepts.

Determining how to help students come to a proper understanding of mathematical proofs and enhancing student proof techniques has been a difficult field in mathematical research. There has been a shift in the research from research looking at ways to promote skills in formal mathematical proofs to how students understanding evolve and in how to best help students improve their understanding (Marrades, 2000).

In order for students to get a basic mastering of proofs, it is essential that students are given the opportunity to work with proofs at a variety of stages of their mathematics education (Franklin, 1996; Hanna, 1995; NCTM, 2000; & Wu, 1996). Davis (2002) states that even though geometry class is traditionally where most students have been introduced to the concepts proof, that this is the wrong place for students to see proofs for the first time. He lists the following as problems: (1) the students are introduced to two new subjects, geometry and proofs at the same time; (2) geometry requires the students to do mathematics that moves away from symbol manipulation that they are accustomed too; and (3) the level of proof tend to be very simple and do not prepare the students for proof in other mathematics classes. Because of this limited exposure to proofs students are not given enough opportunities to encounter proofs in the mathematics classroom, and in order to develop a better student understanding of proofs, students must be given many opportunities to discover and to practice with proofs. Proofs are essential for developing mathematical understanding. When understanding is established, proofs become both credible and valid (Hanna, 2000).

One schematic model called "Conceptual Mathematization" of the learning process developed by De Lang (1987) of how students develop mathematical concepts and ideas from the real world is shown in Figure 1. In this cycle of learning students will come to class with concrete experiences from the real world. Students will then reflect on these experiences and begin to put them in mathematical terms. Through proof the students will be able to justify the abstractions and to better understand the proofs and applications will be generated and this cycle would continue on and on (De Lange, 1996). This model has exploration and proving working together to develop understanding. Exploration leads to discovery, while proof is confirmation. Both of these activities are an essential part of problem solving in mathematics. Exploration can be used to motivate students into finding a proof. It is essential that students learn that exploration and proving is not the same (Hanna, 2000).



Proof and Teachers

In a study conducted by Knuth (2000) involving 18 experienced secondary school mathematics teachers, several roles for proof in secondary school mathematics were suggested. Two of the roles suggested by the teachers spoke towards current mathematical reforms. The first dealt with the role of proof in explaining why a statement is true. Proofs allowed for the children to understand why formulas can be used instead of just accepting a formula to be true because the text says so. Proofs can justify the use of a particular formula.

Another important aspect that teachers identified was the role that proofs had in fostering student independence. For a student be an independent learner of mathematics, they must be able to create their own knowledge and to be able to confirm their own knowledge as well as the knowledge of others. Proofs were thought to be an essential part of the student becoming an independent thinker, thus creating a student that is less dependent upon a single teacher or text. Students would move away from assembly line mathematics were there is no change or original thought required, and move towards becoming creators of knowledge instead of just consumers of knowledge. Proofs therefore are an essential part of the NCTM's Learning Principle which states "Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge."

Although the majority of teachers felt that proofs were an important factor in the learning of mathematics, they still held the belief that formal proofs do not play a central role in secondary mathematics and that it was not appropriate for the majority of students to learn. They felt that only students in advanced classes should study the concepts of proof in mathematics. For the most part they felt that proofs were the domain of the geometry class. This is contradictory to NCTM (2000), which states that proofs must be an essential part of all mathematics curriculums from prekindergarten through grade twelve and to CCSS (2010) which has as a focal point the need for students to be able to reason why the mathematics works with formal formulation of mathematical proofs as they progress from middle to high school.

Teachers did however view informal proofs as an important aspect of the students' mathematics education. For many of the teachers informal proofs allows the students the opportunity to formulate and investigate conjectures and for the students to think about and analyze particular examples. Through the informal proofs students would gain a better understanding of mathematical concepts and be more motivated in the future to explore mathematical concepts more deeply with formal proofs (Knuth, 2000).

Ross (1998) felt that an important role for a teacher of mathematics was the responsibility of explaining the mathematical concepts at the students' level of mathematical knowledge. Students should always be reminded that a logical reason or proof is essential. He felt that proofs should be for there educational value rather than formal correctness. Proofs should be used in a fashion that will lead to understanding or better insight into the mathematical concepts being discussed.

In order for teachers to be prepared to follow the NCTM standards and reform, it will be essential for higher education to engage teachers in classroom experiences with proofs as an important part of their mathematical training. Alibert and Thomas (1991) stated, "[the] context in which students meet proofs in mathematics may greatly influence their perception of the value of proof. By establishing an environment in which students may see and experience first-hand what is necessary for them to convince others, of the truth or falsehood of propositions, proof becomes an instrument of personal value which they will be happier to use [or teach] in the future."

The preparation of teachers to teach mathematics needs to be reexamined. In most cases university mathematicians teach courses to serve professional communities such as engineers, economists, and biologists. When teaching teachers, most universities do not consider how teacher will apply the mathematics in the classroom and how to establish a community of reasoning among kindergarten through grade 12 students. It is essential that kindergarten through grade 12 mathematics teachers learn that they will be involved in the development of children's capacity to construct proofs, to understand the need for justification, and to be able to differentiate between valid justification and invalid justifications (Ball, 2000). In other words, teachers will need to have experiences involving proofs that demonstrate that proofs are an important and meaningful tool that is worth studying and is an essential way of communicating mathematics rather than some exercise that one does to please the mathematics instructor (Knuth, 2000).

It will be important to engage the teachers in discussion so that they can develop for themselves a better understanding of what constitutes a proof. If teachers do not have a proper understanding of what a proof is and its role in mathematics, then it not be surprising that they view proofs as not important (Knuth, 2000).

Conclusion

For the students to change in how they think, teachers must change in the way they teach. Teachers tend to teach in the same fashion that they themselves learned the subject manner for that is human nature. If there is any hope that our students will one day understand what it means to do mathematics, teachers must be able to understand for themselves the subject matter. In order for this to occur, teachers must receive the support and training necessary to make a difference in how students think. Research has found that when teachers receive the proper professional development in mathematics that the students had greater achievement (Kilpatrick, 2002; & U.S. Department of Education, 2000).

Mathematics must be more than just learning facts and skills for computations. The main goal of a mathematics class should be to create an awareness of the what, the how, and the why of mathematics so that the student can understand the mathematics and see how mathematics is part of their lives (Devlin, K).

A challenge for mathematics educators will be getting students to see the relationship between applications, proofs, pure mathematics, and the real world. Mathematics educators will need to learn more on how students approach mathematics and proofs in general. Students should not be expected to master proofs in a single class such as geometry. Proofs and reasoning skills are such that they need to be developed over time throughout a students educational experiences. This way students will be able to start with simple mathematical concepts to reason through and prove and develop their skills so that they may be able to meet the challenges of more complex mathematics that has become an integral part of living in our advanced technological global society.

† *David C. Bramlett, Ph.D.,* Jackson State University, USA
‡ *Carl T. Drake, Ph.D.,* Jackson State University, USA

References

American Mathematical Society Association Resource Group (AMS ARG), (2000). Response to NCTM's round 4 questions. Retrieved June 12, 2003 from the World Wide Web: http://www.ams.org/government/argrpt4.html

Ball, D. L., & Bass, H. (2000). Investigating teaching practice: Setting the stage. Knowing and Learning Mathematics for Teaching: Proceedings of a Workshop.

Common Core State Standards Initiative. (CCSS) (2010). Common core state standards for mathematics. Washington, D.C: National Governors Association Center for Best Practices and the Council of Chief State School Officers.

Davis, T. (2002). A better mathematics curriculum. The Math Forum. Retrieved July 4, 2003 from the World Wide Web: http://mathforum.org/dr.math/faq/better_curriculum.html

Deutsch, A. (2000). Proof – what, why and how? Pittsburg Teachers Institute. Retrieved July 4, 2003 from the World Wide Web: http://www.chatham.edu/PTI/ProofinMathematics/abstracts.htm#proof

De Lang, J. (1996). Using and applying mathematics in education. International Handbook of Mathematics Education, 49 – 97.

Devlin, K. (1998). Why we should reduce skills teaching in the math class. MAA Online. Retrieved June 22, 2003 from the World Wide Web: http://www.maa.org/features/skills.html

Franklin, F. J. (1996). Proof in mathematic education. *Journal of Education*. 178(1) 35-46.

Hanna, G. (2000). Proof, explanation and exploration: an overview. *Educational Studies in Mathematics*. 44, 5 – 23.

Hiebert, J., Carpenter, T. P., Fennema, E., Fusan, K. C., Wearne, D., Murray, H., Olivier, A., & Human, P. (1997). Making sense: Teaching and learning mathematics with understanding. Heinemann.

Herbst, P. G. (2002). Establishing a custom of proving in american school geometry: Evolution of the two column proof in the early twentieth century. *Educational Studies in Mathematics*. 49, 283 – 312.

Kilpatrick, J. (1997, October 4). Five lessons from the new math era. Paper presented at the 1997 Symposium Reflecting on Sputnik: Linking the Past, Present, and Future of Educational Reform. Retrieved June 29, 2003 from http://www.nas.edu/sputnik

Kilpatrick, J., & Swafford, J (2002). *Helping children learn mathematics.* Washington D.C.: National Academy of Science.

Klein, D. (2003). A brief history of american k-12 mathematics education in the 20th century. Mathematical Cognition. Retrieved June 27, 2003 from http://www.csun.edu/~vcmth00m/AHistory.html

Kossack, R. (1998). Why are we learning this? Notices of the American Mathematical Society. 42(12), 1528 – 1534.

Knuth, E. (2000). The rebirth of proof in school mathematics in the United States? *International Newsletter on the Teaching and learning of Mathematical Proof*. Retrieved June 10, 2003 from the World Wide Web: http://www.didactique.imag.fr/preuve/Newslatter/0.../000506ThemeUK.htm

Marrades, R., & Gutierrez, A. (2000). Proofs produced by secondary school students learning geometry in a dynamic computer environment. Educational Studies in Mathematics. 44, 87 – 125.

National Council of Teachers of Mathematics. (NCTM) (1989). Principles and Standards for School Mathematics, Commission on Standards for School Mathematics, Reston, VA.

National Council of Teachers of Mathematics. (NCTM) (2000). Principles and Standards for School Mathematics, Commission on Standards for School Mathematics, Reston, VA.

Ross, K. A. (1998). Doing and proving: The place of algorithms and proofs in school mathematics. *American Mathematical Monthly*, 3, 252 – 255.

U.S. Department of Education. (2000). Before it's too late: A report from the national commission on mathematics and science teaching for the 21st century (Publication No. EE 0449P). Jessup, MD: Author.

Wu, H. (1996). The mathematician and the mathematics education reform. *Notices of the American Mathematical Society*, 43(12), 1531 – 1537.