A Comparison of Geometric Contents

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Abstract

For several categories of geometric figures in the Cartesian plane, necessary and sufficient conditions are established for equality between the area and perimeter associated with each type of figure. In two particular categories, the results between area and perimeter in two dimensions are extended to volume and surface area in three dimensional space. In the same two categories, the above results are then generalized to address equality between n-dimensional and (n–1)-dimensional geometric content for $n \geq 2$.

Introduction

Many curves in the Cartesian plane $\mathbf{R}^2$ are associated with both an area and a length. A real valued function f(x) which is differentiable over a finite interval [a,b] may have an associated area $\int_a^b |f(x)|\,dx$ between the curve y = f(x) and the x-axis over [a,b] as well as an arclength $\int_a^b \sqrt{1+[f'(x)]^2}\,dx$ over [a,b].

For example, suppose that f(x) = cosh(x) for $a \leq x \leq b$. Since cosh(x) > 0 for all real values of x, then the area between the curve y = f(x) and the x-axis is Area = $\int_a^b \cosh(x)\,dx$. On the other hand, the length of the curve y = f(x) is Arclength = $\int_a^b \sqrt{1+[f'(x)]^2}\,dx$ = $\int_a^b \sqrt{1+\sinh^2(x)}\,dx$ = $\int_a^b \sqrt{\cosh^2(x)}\,dx$ = $\int_a^b \cosh(x)\,dx$ (since cosh(x) > 0) = Area. Consequently the area between the curve generated by f(x) = cosh(x) and the x-axis over any interval [a,b] is numerically the same as the length of the curve generated by the same function over the same interval.

In two dimensions, each simple closed curve in $\mathbf{R}^2$ has a corresponding area and perimeter. In three dimensions, geometric shapes in $\mathbf{R}^3$ may have an associated volume and surface area. More generally, for each integer $n \geq 2$, geometric shapes in $\mathbf{R}^n$ may have both an n-dimensional geometric content as well as an (n–1)-dimensional geometric content. Various common geometrical shapes will now be considered in order to explore conditions under which their associated n-dimensional and (n–1)-dimensional geometric contents have the same numerical value.

Cubes

A square in $\mathbf{R}^2$ with edges of length s > 0 has area s^2 and perimeter 4s. Therefore its area is numerically equal to its perimeter if and only if $s^2 = 4s$ if and only if s = 4 (since s > 0).

Increasing the dimension by one, a cube in $\mathbf{R}^3$ with edges of length s > 0 has volume s^3 and surface area $6s^2$. Therefore its volume is numerically equal to its surface area if and only if $s^3 = 6s^2$ if and only if s = 6 (since s > 0).

The last two special cases can be generalized as follows. If n is an integer and n ≥ 2, then an n-cube in $\mathbf{R}^n$ with edges of length s > 0 has n-dimensional geometric content s^n and (n–1)-dimensional geometric content $2ns^{n-1}$ [2, p. 15]. As a result, the n-dimensional and (n–1)-dimensional geometric contents are numerically equal if and only if $s^n = 2ns^{n-1}$ if and only if s = 2n (since s > 0).

Spheres

A circle in $\mathbf{R}^2$ with radius r > 0 has area πr^2 and circumference 2πr. Thus the area is numerically equal to the circumference if and only if $\pi r^2 = 2\pi r$ if and only if r = 2 (since r > 0).

Furthermore, a sphere in $\mathbf{R}^3$ with radius r > 0 has volume $\frac{4}{3}\pi r^3$ and surface area $4\pi r^2$. Thus the volume is numerically equal to the surface area if and only if $\frac{4}{3}\pi r^3 = 4\pi r^2$ if and only if r = 3 (since r > 0).

Similar to the results for cubes, the cases for spheres in $\mathbf{R}^2$ and $\mathbf{R}^3$ can be extended as follows. If n is an integer and n ≥ 2, then an n-sphere in $\mathbf{R}^n$ of radius r > 0 has n-dimensional geometric content $\dfrac{2\pi^{\frac{n}{2}} r^n}{n\Gamma\left(\frac{n}{2}\right)}$ [4, p. 35, formula (110)] and (n–1)-dimensional geometric content $\dfrac{2\pi^{\frac{n}{2}} r^{n-1}}{\Gamma\left(\frac{n}{2}\right)}$ [4, p. 36, formula (111)]. Therefore the n-dimensional geometric content is numerically equal to the (n–1)-dimensional geometric content if and only if $\frac{2\pi^{\frac{n}{2}} r^n}{n\Gamma\left(\frac{n}{2}\right)} = \frac{2\pi^{\frac{n}{2}} r^{n-1}}{\Gamma\left(\frac{n}{2}\right)}$ if and only if $r^n = nr^{n-1}$ if and only if r = n (since r > 0).

Triangles

Consider a general multiple 3r-4r-5r (r > 0) of the 3-4-5 right triangle. Such a triangle has area $\frac{1}{2}(3r)(4r) = 6r^2$ and perimeter 3r + 4r + 5r = 12r. Therefore the area and perimeter are numerically equal if and only if $6r^2 = 12r$ if and only if r = 2 (since r > 0). Consequently the 6-8-10 right triangle is the unique multiple of the 3-4-5 right triangle whose area is numerically equal to its perimeter.

As another example, consider an arbitrary multiple 5r-12r-13r (r > 0) of the 5-12-13 right triangle. This triangle has area $\frac{1}{2}(5r)(12r) = 30r^2$ and perimeter 5r + 12r + 13r = 30r. Therefore the area and perimeter are numerically equal if and only if $30r^2 = 30r$ if and only if r = 1 (since r > 0). Consequently the 5-12-13 right triangle itself is its own unique multiple whose area and perimeter are numerically equal.

An equilateral triangle with sides of length s > 0 clearly has perimeter 3s. Furthermore, its area is $\frac{1}{2}(s)(s \cdot \sin 60°) = \frac{\sqrt{3}}{4}s^2$. Therefore its area and perimeter are numerically equal if and only if $\frac{\sqrt{3}}{4}s^2 = 3s$ if and only if s = $4\sqrt{3}$ (since s > 0). Hence the only equilateral triangle with the property that its area is numerically equal to its perimeter is the one whose sides each have length $4\sqrt{3}$ units.

Regular Polygons

Note that the preceding case of the equilateral triangle can alternately be described as a regular 3-sided polygon. In a similar manner, the case of the square above is a regular 4-sided polygon. We now generalize these cases by considering arbitrary n-sided regular polygons. To this end, suppose that n is an integer, n ≥ 3, s is a real number, and s > 0. For notational purposes we define $P_{n,s}$ to be a regular polygon with n sides, each of length s. We further define A(n,s) to be the area enclosed by $P_{n,s}$ and P(n,s) to be the perimeter of $P_{n,s}$.

Clearly $P_{n,s}$ has perimeter P(n,s) = ns. In order to develop a formula for the area of $P_{n,s}$, suppose that $P_{n,s}$ has center C, apothem r, and adjacent vertices A and B. (See Figure 1 below.)

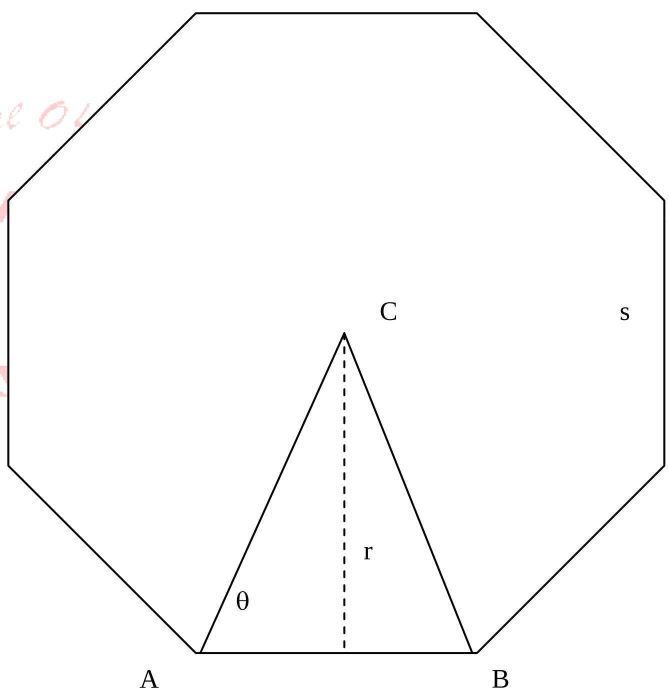

Figure 1

Then each interior angle of $P_{n,s}$ has measure $\frac{n-2}{n}\pi$ radians ([1, p. 99, Corollary 2.5.3],[3, p. 242, Corollary A],[5, p. 128, Corollary 4.17]). Furthermore, the segment between C and A bisects the interior angle at vertex A [1, p. 341, Theorem 7.3.3]. (See Figure 1 above.) Therefore $\theta = \frac{1}{2}\cdot\frac{n-2}{n}\pi = \frac{n-2}{n}\cdot\frac{\pi}{2}$. Finally, $r = \frac{s}{2}\tan\theta$, so that $r = \frac{1}{2}s\cdot\tan\left(\frac{n-2}{n}\cdot\frac{\pi}{2}\right)$. (1)

The area of ΔABC is $\frac{1}{2}sr$. Furthermore, the area A(n,s) of $P_{n,s}$ consists of n such areas. Thus $A(n,s) = n\left(\frac{1}{2}sr\right) = \frac{1}{2}r(ns)$. Consequently we obtain the well known result ([1, p. 342, Theorem 7.3.5],[3, p. 86, no. 13]) that

$$A(n,s) = \frac{1}{2} r \cdot P(n,s)\,. \tag{2}$$

From (2) we see that A(n,s) = P(n,s) if and only if r = 2 if and only if $\frac{1}{2} s \cdot \tan\left(\frac{n-2}{n} \cdot \frac{\pi}{2}\right) = 2$ (from (1) above) if and only if $s = \dfrac{4}{\tan\left(\frac{n-2}{n} \cdot \frac{\pi}{2}\right)}$. Thus the area of $P_{n,s}$ is numerically equal to its perimeter if and only if

$$s = 4\cot\left(\frac{n-2}{n} \cdot \frac{\pi}{2}\right). \tag{3}$$

It is noteworthy that for n = 3, $P_{n,s}$ is an equilateral triangle. In this case the result in (3) is consistent with the result above for equilateral triangles. Furthermore, $P_{n,s}$ is a square for n = 4. In this case the result in (3) is also consistent with the result derived above for the cube with n = 2.

Note also that for each integer $n \geq 3$, $0 < \frac{n-2}{n} \cdot \frac{\pi}{2} < \frac{\pi}{2}$, so that $\cot\left(\frac{n-2}{n} \cdot \frac{\pi}{2}\right) > 0$. Thus from (3) above we have $s = 4\cot\left(\frac{n-2}{n} \cdot \frac{\pi}{2}\right) > 0$. Furthermore, the function $s(\alpha) = 4\cot\alpha$ is 1-1 for $0 < \alpha < \frac{\pi}{2}$. Hence for each integer $n \geq 3$, there is a *unique* regular polygon with n sides whose area is numerically equal to its perimeter. More specifically, the unique regular n-sided polygon $P_{n,s}$ whose area and perimeter are numerically equal is the one whose sides have length s satisfying (3) above.

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References

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