

A History of Mathematical Proof: Ancient Greece to the Computer Age

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Abstract

This paper will examine the evolution of proof in mathematics throughout time. The concept of mathematical proof had its beginnings with the ancient Greeks. The paper will start with Thales of Miletus, who was given credit for the first mathematical proof, and follow the evolution of proof through the high point of Greek mathematics with Euclidean Geometry, 17th and 18th century return to mathematics, and the return of rigor and the axiomatic method in the 19th and 20th century.

Introduction

Mathematics as we know it today had its beginnings when the ancient Greeks would take the knowledge of the Egyptians and Babylonians and make it into their own. Mathematics would no longer just be a tool to solve some practical application, but instead become a way of thought. The Greeks would be the first to transform mathematics into a logical method of viewing the world. A major component would be the concept of proof, which would allow them to distinguish with logic the difference between what can and cannot be done. Grabiner (1974) stated that the Greeks would transform mathematics from an experimental science to an intellectual science. It would be the Greeks that would be the first to transform mathematical statements through logical arguments. Deductive methods would again be used in the seventeenth and eighteenth century with the introduction of calculus and continuous functions. In the eighteenth and nineteenth century, mathematics would again be transformed with the discoveries of calculus and the discovery of non-Euclidean geometries. By the mid 1800s mathematics would become elaborate and abstract and a more rigid formal structure would find its way into the concept of proof.

Non-Euclidean geometry of the 1820s, various forms of abstract algebra in the mid 1800s, and the transfinite numbers of the 1880s would move mathematics away from any obvious connections to everyday life and towards a more abstract approach in mathematics. At the end of the 1800s David Hilbert emphasized that all mathematics could be derived by starting from axioms and using the formal process of proof (Wolfram, 2002).

Mathematical proof in basic terms is simply the means of convincing someone or oneself that something is true using an argument based on reason. Bell (as cited in Almeida, 1994) gives a more precise interpretation of proof, which goes as follows; "A proof is a directed tree of statements, connected by implications, whose end point is the conclusion and whose starting points are either in the data or are generally agreed facts or principles" (p. 661).

The evolution of the type of proofs that mathematicians use are proof by deduction, by mathematical induction, by transfinite induction, by exhaustion, by contradiction, and by construction.

The Greeks and the Birth of Proof

Although more ancient societies had used mathematics, it would be the Greeks that would change how science and mathematics would be viewed. It would all begin when the Ionian Greeks began to answer the why questions of the universe. The Greeks that would be known as the Ionians emigrated from mainland Greece in about 1000 BC eastward to Ionia and settled on the islands and the coastline of Asia Minor. It would not be enough to just know how something works for they also wanted to know why something works.

The concept of proof in mathematics would have its foundation with the concept of deductive reasoning. The Greeks would apply deductive reasoning to all aspects of thought. Deductive reasoning operates on the premise there are some known or assumed factors that are true. Through deductive reasoning, these facts are then used to discover new facts. Through deductive reasoning known premises are known or assumed from which a new fact or conclusion follows. An early example of deductive reasoning goes as follows: All men are mortal, Socrates is a man, and therefore Socrates is mortal.

The Ionians would take the geometry that the Egyptians used for building the pyramids and make it a tool for practical application (Burke, 1995). Geometry would become for all Greeks the essential tool to view and understand the world around them. They would also take the time to record their discoveries for future generations to build upon. Mathematics would so engulf their society that it was almost religious in nature for their philosophy of life and mathematics went hand in hand.

By the time of Pythagoras and the Pythagoreans, they believed that the gods made numbers the basis of world order. They believed that the divine harmony of the universe could be seen by numerical relationships (Turchin, 1977). The discovery that pleasant sounds were created by forming the whole number ratios such as 1:2 (octave), 2:3 (fifth), 3:4 (fourth), and so forth would drive their belief that all natural phenomena can be derived from whole numbers. The Pythagoreans came to believe that numbers were the atoms that made up the universe. Clawson (1996) writes that even the Pythagoreans' version of the creation story centered on "The One" which was a monad without differentiation or extension. The monad was surrounded by what was referred to as the unlimited. The unlimited separated the monad into individual atomic numbers that would reorganize themselves into various geometric shapes, which would become the four elements earth, air, fire, and water. Within this universe existed harmony so that all elements of the universe were in correct whole number ratios. This religious zeal for mathematics would both drive their thirst for new discoveries and proofs and also limit what could be discovered and proved (Dunham, 1990).

Mathematics would move away from the strictly practical applications of the ancient Egyptian and Babylonian societies and begin its journey toward the abstract and philosophical nature of mathematics. The Greeks were the first known society to try and prove mathematical concepts through mathematical reasoning (Kleiner, 1991).

Thales and the First Proof

Thales of Miletus (634-548 B.C.) is the first person to be given credit for discoveries made in mathematics. A Thales early contribution to Ionian society was in the use of astronomy for maritime navigation. He would also establish a school in the Greek town of Miletus. Miletus was a trading town in the west coast of Asia Minor that was involved with trading with the ancient civilizations of Egypt and Babylonia. Thales was considered by later generations of Greeks as one of the seven wise men of Greece.

His greatest contribution to mathematics was his use of deductive reasoning for finding new mathematical truths. He would use known truths about mathematics to deduce new truths. The Greeks would adopt this as the centerpiece of Greek thinking and this would become the prominent characteristic of Greek mathematics (Clawson, 1996). The first proof in the history of mathematics is considered to be when Thales proved that the diameter of a circle divides a circle into two equal parts. This is the earliest known recorded attempt at proving mathematical concepts.

Thales early life as a merchant provided him with enough wealth to be able to devote his time to study and to travel to ancient civilizations such as Egypt. In Egypt, he would use mathematics to deduce a way to measure the height of a pyramid using the relation of similar triangles. Thales like many of the Greeks used the knowledge that was established by the Egyptians and Babylonian societies to seek out proof. Eves (1963) list some other results in which Thales is given credited for are as follows:

- 1) A circle is bisected by any diameter.
- 2) The base angles of an isosceles triangle are equal.
- 3) The vertical angles formed by two intersecting lines are equal.
- 4) Two triangles are congruent if they have two angles and one side in each respectively equal.
- 5) Any angle inscribed in a semicircle is a right angle.

Thales is also considered by some historians to be the first known mathematician to have constructed a circle circumscribed about a right triangle. This discovery so excited Thales that he sacrificed an ox to celebrate and give thanks for his accomplishment. Turchin (1977) would describe Thales proof in the following way:

Draw a right triangle ABC. Divide hypotenuse BC into two equal segments at point D. Connect D to Point A. If $AD = DC = BD$, we can draw a circle by having D as its center and AD as its radius. This circle will then pass through points A, B, and C. Add point E creating a rectangle ABEC and draw diagonal DE. The proof is now completed by the following reasoning: triangle ABC and AEC are equal because

they have side AC in common, sides AB and EC are equal and angles BAC and ECA are right angles and angle EAC is equal to angle BCA. That is triangle ADC is an isosceles triangle.

Greek proofs of this time period and afterwards relied heavily on the verbal tradition and a tradition of construction (p. 214 – 215).

Turchin (1977) and Davis (1981) both stated that that the simplicity of these basic theorems and their intuitive obviousness demonstrates that Thales knew the importance of proof. These theorems were not proved because there was any doubt in their truth, but instead he was trying to develop a systematic technique to prove an idea. Another aspect of the early proofs was that they were very verbal for the symbolism that we are familiar with has not been developed in this time in history.

Mathematics would forever be changed with the birth of the concept of proof in mathematics. With Thales the seeds of proof in mathematics had been planted for the future generations of mathematics to harvest.

The mathematics of the ancient Eastern countries had virtually remained unchained for two millennia. The Greeks would in one to two centuries create almost all the geometry that is still in use today. The evolution of the mathematics of this period begins with Thales and reaches its greatest heights with Euclid's famous text Elements that would influence countless mathematicians for generations to come.

The Pythagoreans

One of the best-known proofs that would come out of the Greek Age would be the famous Pythagorean Theorem that all school children still learn today. Ironically some believe that this famous proof was not actually proven rigorously by Pythagoras. On the other hand a more controversial proof would be accredited to a Pythagorean by the name of Hippasus of Mesopotam. He would show that $\sqrt{2}$ is an irrational number. This proof would so enrage the Pythagoreans that it is said that they had him drowned for there belief was that the entire universe could be reduced to whole numbers and their ratios (Dunham, 1990). The proof of $\sqrt{2}$ is an excellent example of a proof by contradiction. Walthoe (1999) gives this proof for the $\sqrt{2}$ as being an irrational number:

Assume that $\sqrt{2}$ is rational. Then we can find whole numbers A and B such that,

$$\sqrt{2} = \frac{A}{B},$$

and A and B have no common factors. From this it follows that

$$A^2 = 2B^2$$

Thus A^2 is even, and since the squares of the odd numbers are always odd, we can deduce that A is also even. So, $A = 2a$ where a is also a whole number, and re-writing the above line gives $2a^2 = B^2$.

We now know that B is also even and thus $B = 2b$. This means that 2 divides both A and B as such is a common factor. This contradicts our hypothesis, which then cannot be true. From this it follows that $\sqrt{2}$ is irrational (p. 2-3).

Euclidean Geometry

Euclid's works represent some of the earliest use of mathematical proofs that have remained in use for over 2000 years. Euclid laid the groundwork for future mathematics by organizing the known mathematics into definitions, assumptions, and postulates.

In his most famous work, Elements, he began with twenty-three definitions based on lines, points, circles, and various other concepts, ten assumptions, and five postulates. The postulates represented the axioms in Euclid's geometry from which through deductive reasoning he would prove other concepts in mathematics. The axioms are logical beliefs that are assumed to be true rather than proven. Euclid's method of proof would become known as the Axiomatic Method. Proofs now had a more formal foundation upon which to build upon.

With Euclid the verbal and constructive tradition of proving would become more rigorous for each step of the process of the proof could now be justified by some definition, assumption, or axiom. A good example of an axiomatic treatment of a proof would be Euclid's proof of his Proposition 45. I quote it from the English version given by Sir Thomas L. Heath (1956). The in-text numbers to the right of the statements are Euclid's references to previously established results, definitions, assumptions, or axioms.

Proposition 45: To construct, in a given rectilineal angle, a parallelogram equal to a given rectilineal figure. Let $ABCD$ be the given rectilineal figure and E the given rectilineal angle; thus it is required to construct, in the given angle E , a parallelogram equal to the rectilineal figure $ABCD$.

Let DB be joined, and let the parallelogram FH be constructed equal to the triangle ABD , in the angle HKF which is equal to E ; [I. 42]

Let the parallelogram GM equal to the triangle DBC be applied to the straight line GH , in the angle GHM which is equal to E . [I. 44]

Then since the angle E is equal to each of the angles HKF , GHM , the angle HKF is equal to the angle GHM . [C. N. I]

Let the angle KHG be added to each; therefore the angles FKH , KHG are equal to the angles KHG , GHM . But the angles FKH , KHG are equal to two right angles; [I. 29]

Therefore the angles KHG , GHM are also equal to two right angles. Thus, with a straight line GH , and at the point H on it, two straight lines KH , HM not lying on the same side make the adjacent angles equal to right angles; therefore KH is in a straight line with HM . [I. 14]

And, since the straight line HG falls upon the parallels KM , FG , the alternate angles MHG , HGF are equal to one another. [I. 29]

Let the angle HGL be added to each; therefore the angles MHG, HGL are equal to angles HGF, HGL. [C. N. 2]
 But the angles MHG, HGL are equal to two right angles; [I. 29]
 Therefore the angles HGF, HGL are also equal to two right angles.[C. N. 1]
 Therefore FG is in a straight line with GL. [I. 14]
 And, since FK is equal and parallel to HG, [I. 34]
 And HG to ML also,
 KF is also equal and parallel to ML; [C. N. 1; I. 30]
 And the straight lines KM, FL join them (at their extremities);
 Therefore KM, FL are also equal and parallel; [I. 33]
 Therefore KFLM is a parallelogram. And since the triangle ABD is equal to the parallelogram FH, and DBC to GM, the whole rectilinear figure ABCD is equal to the whole parallelogram KFLM.
 Therefore the parallelogram KFLM has been constructed equal to the given rectilinear figure ABCD, in the angle FKM which is equal to the given angle E. Q.E.D. (p. 345-347)

Euclid set the tradition and methodology that mathematicians for centuries to follow would use as their basis of proof. Many people would continue to be amazed and sometimes even in disbelief in how Euclid built upon the past to discover the future. John Aubrey in his book "Brief Lives" as quoted in Davis (1995) describes his reaction to Euclid's Proposition 47 and Euclidian Geometry as follows:

He was 40 years old before he looked on Geometry; which happened accidentally. Being in a Gentleman's Library, Euclid's Elements lay open, and 'twas the 47 El *libri* I. He read the Proposition. By G_, sayd he (he would now and then sweare an empatical Oath by the way of emphasis) *this is impossible!* So he reads the Demonstration of it, which referred him back to such a Proposition; which proposition he read. That referred him back to another, which he also read. *Et sic deinceps* [and so on] that at last he was demonstratively convinced of the truth. This made in him love with Geometry (p.164-165).

With Euclid mathematics had a formal structure on which new mathematics could be discovered.

The Decline of Greek Mathematics

Apollonius would be the last great Greek mathematician. The work that Apollonius did with conic sections would remain unchanged until Descartes in the seventeenth century. There are many reasons given for the decline in Greek mathematics such as political and social unrest of the times, but the main reason that Turchin (1977) gives is the Greeks failure to create algebraic symbols. The Greeks emphasis on verbal proofs and constructive proofs eventually limited them to only certain areas of mathematics.

17th and 18th Century Renaissance of Mathematics and Intuition

The next great evolution of mathematics after the Greeks would come in the late 17th century with mathematicians like Marin Mersenne, Rene Descartes, and Isaac Newton. The centuries that followed the Greek Age was one of little development in the theory of proof. Although Arithmetic and Algebra would continue to flourish with Hindu and Arab mathematicians, rigorous proof would not be of great consequence. The renaissance marked the end of the Dark Ages and the beginning for the quest for new knowledge in Europe. For the first time since the ancient Greeks, people would be able to seek knowledge for the sake of knowledge.

Europe would experience a cultural reawakening which would result in the cultural and scientific revolution that would occur during this time period. During the Renaissance finding new knowledge had become the primary purpose of all sciences. The first newly recorded result in mathematics would be the discovery of the solution to the cubic equation in 1545. With this discovery, results would become the primary drive for mathematicians. As a consequence, the rigor of proofs would often be less than rigorous. Mathematicians would come to rely more on their intuition than in proving mathematics in the axiomatic way. Calculus would come to dominate this age for it produced many wonderful results even though the mathematician of this day could not necessary explain why it worked (Grabiner, 1974).

Johannes Kepler (1571 – 1630) used mathematics to find the paths of planets. He attempted to make his proofs as sound as possible by spelling out each step in detail and emphasizing the physical basis of his mathematical procedures (Field, 1999). They were still mathematically rigorous and introduced the concept of observational error.

Kepler would influence mathematicians and scientist of his time and his work would be an important developmental stage for proof in the area of science. Kepler's preferred method of proof still relied heavily on Euclid for he started with definitions, then axioms, and then proceeded with deductions. Problems occurred for the deductive method did not do well with messy numbers from countless observations. Often his results would be considered true simply by a method of induction. Induction in this case means that if a statement is true in enough special cases, then one could determine that it must be true in all cases (Walthoe, 1999). By today's standards his proofs would not be considered proofs for they were based on collecting data and a haphazard method of induction. For example Walthoe (1999) considered the following formula, $P(n) = n^2 + n + 41, n \in W$, which is a formula that seems to generate prime numbers. We can see for that $P(0) = 41, P(1) = 43, P(2) = 47, P(3) = 53, P(4) = 61, P(5) = 71$ are all prime and there are many more primes that follow. It would seem then $P(n)$ does indeed generate prime numbers, but $P(40) = 1681 = (40)(40)$ is a composite number (p. 3-4). This is a perfect demonstration that just because a statement is true in enough special cases, that it will not be true for all cases. This would be a problem with many proofs of this time period.

The Rise of Symbolic Notation

With the great need for results came the need for a symbolism system to make computations more convenient. The use of symbolism led to many successful results in both algebra and calculus. Much of the general symbols that we still use today were first introduced in 1591 by French mathematician Francois Viète (Grabiner, 1974). Symbolism would now become a more essential part of mathematical proof. Symbolic notation would become an important tool not only for demonstration and as a pedagogical aid but would also become an important part of discovery in mathematics.

For example if we consider the polynomial $(x-a)(x-b)(x-c) = x^3 - (a+b+c)x^2 + (ab+ac+bc)x - abc$. From the symbolic notation, 17th and 18th mathematicians could discover the relation between the roots and the coefficients of any polynomial equation of any degree. In this equation one can deduce that the equation has degree 3 and has three roots. The result of studying problems of this nature would result in Albert Girard in 1629 stating that an equation of degree n had n roots (Grabiner, 1974 & Kleiner, 1991). This would be the basis for Gauss's Fundamental Theorem of Algebra.

The use of symbolic notation would result in proofs being easier to demonstrate. Cardano's three page proof for the formula of the general solution of a cubic could now be compressed into a half-page proof. The use of symbolic notation would allow mathematicians to explore past the works of the ancient Greeks for the use of symbolic notation made proofs and mathematical concepts more assessable. C. H. Edwards as quoted in Kleiner (1991) had this to say about Leibniz's symbolic notation for calculus, "It is hardly an exaggeration to say that the calculus of Leibniz brings within the range of an ordinary student problems that once required the ingenuity of an Archimedes or a Newton (p. 294)." The success of Leibniz's notation such as $\frac{dy}{dx}$ and $\int y dx$ would help to reinforce the mathematicians' beliefs in the power of symbolism to yield true conclusions (Grabiner, 1974).

Leonhard Euler's work with symbolic notation would be some of the greatest of his time. Kleiner (1991) describes his proofs as grand art. He was truly a master at the manipulation of symbols within a proof. Grabiner (1974) and Kleiner (1991) both describe Euler's proof of the infinite series for the cosine of the angle as follows:

Euler began with the identity

$$(\cos z + i \sin z)^n = \cos nz + i \sin nz .$$

Now using the binomial theorem, expand the left-hand side and equate the real part to $\cos nz$ to obtain,

$$\cos nz = (\cos z)^n - \frac{n(n-1)}{2!}(\cos z)^{n-2}(\sin z)^2 + \frac{n(n-1)(n-2)(n-3)}{4!}(\cos z)^{n-4}(\sin z)^4 - \dots$$

Now letting n be an infinitely large integer and z an infinitely small number. Then

$$\cos z = 1, \quad \sin z = z, \quad n(n-1) = n^2, \quad n(n-1)(n-2)(n-3) = n^4, \dots$$

The equation now can be rewritten as

$$\cos nz = 1 - \frac{n^2 z^2}{2!} + \frac{n^4 z^4}{4!} - \dots$$

Because n is infinitely large and z is infinitely small Euler concluded that nz must be a finite quantity such that $nz = x$ and we get the following:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \quad (\text{p. 355 \& p. 295}).$$

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Algebraic analysis of this sort would be how most 18th century mathematicians would approach a proof. The use of symbolic notation and the acceptance of many assumptions without any rigorous proof such as what is true for convergent series is true for divergent series, what is true for finite amounts is true for infinitely small and large amounts, and what is true for polynomials is true for power series would drive the mathematics of the 18th century (Grabiner, 1974 & Kleiner, 1981).

Education

Logical Fallacy of the 18th Century

By the end of the 18th century it had become apparent that the constant strive for results led to many inconsistencies and many questions to be answered. One of the greatest critics of 18th century was George Berkeley (1685 – 1753) who felt that calculus was filled with logical fallacy. Berkeley's publication of "The Analyst" includes point by point criticisms of some of the main arguments found in Newton's calculus (Grabiner, 1974). Eves (1981) uses Newton's Quadrature of Curves of 1704 as an example. In determining the derivative of x^3 , Newton essentially did the following:

In the same time x , by growing, becomes $x + o$, the power x^3 becomes $(x + o)^3$, or

$$x^3 + 3x^2o + 3xo^2 + o^3,$$

and the growths, or increments,

$$o \text{ and } 3x^2o + 3xo^2 + o^3$$

are to each other as

$$1 \text{ to } 3x^2 + 3xo + o^2 \quad (\text{p. 133}).$$

The problem that Berkeley had was the shift in hypothesis that occurred in this proof and in others of this time period. In one part o is considered as non-vanishing while in another part it is considered to be zero. Mathematicians of the day could not properly defend against the criticisms without turning to a rigorous treatment of limits and proofs that would not occur until the 19th century. Joseph Louis LaGrange (1736 – 1813) would be one the first

mathematicians to attempt to put rigor in the proofs of calculus by representing functions using Taylor expansions. With his attempts marks the beginning of the move away from intuition and blind formal manipulation of proofs in analysis (Eves, 1981).

19th Century Mathematics: The Return of Rigor

The greatest changes in mathematics since Euclid and the ancient Greeks would begin in the 19th century. The 19th century will see the introduction of non-Euclidean geometry, followed by many forms of abstract algebra, and transfinite numbers. Within this century mathematics would make a great leap into abstraction and the development of mathematics subject matter. It would be shown by the end of the 19th century that mathematics could be done using only abstract structures which had no connections to everyday intuition. Set Theory would end this century and begin the next as the foundations of all mathematics (Wolfram, 2002).

Another important aspect that would mark the 19th century would be the return of rigorous proof. Non-Euclidean Geometry and the developments of Quaternion numbers that did not follow the normal laws of arithmetic forced mathematicians to have some doubts in the rules of arithmetic and geometry. It would become evident that many of the previous proofs were for the most part based on intuition and needed to be reexamined. More formal definitions and proofs for irrational numbers, continuity, derivatives, integrals, and other mathematical concepts would occur within this century resulting in algebra becoming more popular than geometry as the tool of choice for proofs. The formal structure of mathematics would mean that mathematicians could no longer just justify their work based on intuition (Walthoe, 1999).

The result of this new vigor in proof in mathematics resulted in mathematics once again turning back to the study of axioms. With the axiomatization of mathematics came the development of the foundation for both old and new mathematics.

It would be Carl Frederick Gauss (1777 – 1855) who would set the new standards of mathematical rigor. With his break from the intuitive argument with his 1812 publication on hypergeometric series, Gauss is considered to be the first mathematician to give adequate consideration of the convergence of an infinite series (Eves, 1981).

Augustine-Louis Cauchy (1789 – 1857) considered by some as the French counterpart to Gauss would bring rigor to calculus. In 1821 he would develop an acceptable theory of limits from which he would define continuity, differentiation, and the definite integral. His rigor would inspire others to rid analysis of intuitive reasoning and informal manipulation.

Another key contributor of rigor would be Karl Weierstrass (1815 – 1897). Weierstrass pushed for a mathematical foundation that began with logical development of the real numbers, then the limit concept, continuity, differentiability, convergence, and divergence all defined in terms of the number

system. It would be his work that would result in the “epsilon-delta” definition of limits which goes as follows:

If, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ when $0 < |x - c| < \delta$, then we say L is the Limit of $f(x)$ as $x \rightarrow c$.

Eves (1981) would have this to say about Weierstrass’s accomplishments and the efforts for rigor in mathematics:

Gone are all such phrases as “successive values”, “ultimate ratios”, “taken as small as one wishes”, and “approaches indefinitely close to”. Gone are all the references to growing magnitudes and moving points and the abandonment of infinitely small quantities of higher order. All that remains in this precise and unambiguous language and symbolism are real numbers, the operation of addition (and its inverse, subtraction), and the relationship “less than” (and its inverse, “greater than”) (p. 139).

Throughout the 19th century rigor steadily increased and by the end of the century came the reemergence of the Axiomatic Method. As the 20th century approached, it would give way once again to the rebirth of the Axiomatic Method which would become a distinctive feature of 20th century mathematics (Kleiner, 1991).

20th Century Proof

In the early 20th century the axiomatic method was well established in a variety of mathematical areas. In algebra, major works could be found in group theory (1904), field theory (1910), and ring theory (1914). In analysis, axiomatic formulations could be found in function spaces (1906), general analysis, Banach spaces (1922), Hilbert space (1929), and Hausdoff topological space (1914). Geometry was impacted by Hilbert’s Foundation of Geometry (1899) and Veblen and Young’s abstract treatment on projective geometry (1910 – 1919). Set theory had Zermolo’s axiomatization of set theory (1908), Fraenkel’s improvements (1921), von Neumann’s version (1925), and Russel and Whitehead’s three-volume Principia Mathematica (1910 – 1913) (Kleiner, 1991).

Bourbaki in Kleiner(1991) gives a wonderful description of the axiomatic method of the first part of the 20th century:

What the axiomatic method sets as its essential aim, is exactly that which formalism by itself cannot supply, namely the profound intelligibility of mathematics. Just as the experimental method starts from the *a priori* belief in the permanence of natural laws, so the axiomatic method has its cornerstone in the conviction that, not only is mathematics not a randomly developing concatenation of syllogisms, but neither is it a collection of more or less “astute” tricks, arrived at by lucky combinations, in which purely technical cleverness wins the day. Where the superficial observer sees only two, or several, quite distinct

theories, lending one another “unexpected support” through the intervention of a mathematician of genius, the axiomatic method teaches us to look for deeper-lying reasons for such a discovery, to find the common ideas of these theories, buried under the accumulation of details properly belonging to each of them, to bring these ideas forward and to put them in their proper light (p. 303).

One problem that would occur with proofs in the late 20th century is the problem of proofs simply getting too long. Some proofs would span 400 pages which makes it very difficult to verify with any adequate procedure. Kolata (1976) describes a situation in which one mathematician comes with a proof of a statement and another mathematician comes up with a proof of its negation. They exchange proofs and each mathematician can not find any error in the others works. A third party then reads the two and claims one is correct. The problem is that both proofs are very long and very complicated making it very difficult to verify with any accuracy.

Both Paul Erdos and Ronald Graham believe that the length of many proofs have reached the limit in which a human mind can handle. Some mathematicians such as Michael Rabin of the Hebrew University in Jerusalem have expressed a desire to reduce the definition of proof and allow computers to aid in the proofs allowing for a very low probability error (Kolata, 1976).

Computer Age

A popular proof that is attributed to the computer is the 1976 resolution of the 1850 four-colour conjecture of topology which stated that any map or sphere needs at most four colors to color it so that no two countries sharing a common boundary will have the same color. Kenneth Appel and Wolfgang Haken of the University of Illinois established the conjecture using a computer analysis.

The actual type of proof that they would use would be one of contradiction. They assumed that there existed a map which needed five colors and proceeded to show that this would lead to a contradiction. The final proof consisted of several hundred pages and took over one thousand hours of computer time. Over 2000 possible cases were examined with close to one billion logical options to verify reducibility. The main problem with this type of proof is indeed its length. Many mathematicians still do not believe that this computer result is an actual proof of the four-colored conjecture for it basically impossible to ever verify the entire proof within ones lifetime (Albers, 1981, Eves, 1990, Hunt, 2000, & MacKenzie, 1999). In an address to the American Mathematical Association in 1990, R. Hersh quoted the Halmos’s objections of the four-colour proof:

I do not find it easy to say what we have learned from all that. We are still far from having a good proof of the Four-Colour Theorem. I hope as an article of faith that the computer missed the right concept and the right approach. 100 years from now the map theorem will be, I think,

an exercise in a first-year graduate course, provable in a couple of pages by means of the appropriate concepts, which will be completely familiar by then. The present proof relies in effect on an Oracle, and I say down with Oracles! They are not mathematics (p. 663).

This proof was a far cry of a short and elegant proof that the 19th century mathematicians strived to obtain.

Conclusion

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Since the dawn of proof with the ancient Greeks, determining what constitutes a proof has been an ongoing debate among mathematicians. Rigor in proof has and still is an issue in the presentation of mathematical concepts and ideas. It is a fine balance that one must obtain to construct a proof that is sound and correct while also being understandable by others.

Another constant issue is the fight for ever increasing results which can lead to proofs that may be deemed less than rigorous. The difficulties of some proofs have also lead more and more mathematicians to turn to the use of computers which have lead to problems of verification. I feel that Hersh as quoted in Almeida (1996) best summarizes the notion of proof:

Our inherited notion of 'rigorous proof' is not carved in marble. People will modify that notion, will allow machine computation, numerical evidence, probabilistic algorithms, if they find it advantages to do so. Then we are misleading our pupils, if in the classroom we treat 'rigorous proof' as shibboleth (p. 663).

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