Monoids of Partitions of a Multiset

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Abstract: This research note describes certain monoids consisting of partitions of a multiset.

0.1 Introduction

It is well known that certain monoids consisting of partitions of a set have useful applications in the areas of computer arithmetic, formal languages and sequential machines (see [4] and [11] for example). Taking into account a perceptive use of multisets and multiset-like structures in the recent years (see [1] and [9] for example), this paper intends to describe certain monoids consisting of partitions of a multiset. In section 1, the paper briefly introduces the concept of a multiset, multiset relations and multiset partitions. In section 2, certain monoids of multiset partitions are discribed.

1 Multiset, Multiset Relations and Multiset Partitions

1.1 The Concept of a Multiset

A *bag* or *multiset* (mset for short) A on a set X (called the *base* or the *ground* set) is an unordered collection of objects of X in which duplicates or *multiples* of objects are admitted. In other words an mset is a collection in which objects may appear more than once. Each individual occurrence of an object in an mset is called its element. All duplicates of an object occurring in an mset are treated indistinguishably. The objects of an mset are its distinguishable or distinct elements. Similar to denoting a set, an mset can be denoted by capital or small letters of the English alphabet. The number of occurrences of an object x in an mset A, called its *multiplicity* or *charateristic value* denoted by $m_A(x)$ or $c_A(x)$ or simply A(x) is usually considered finite. The number of distinct elements (which need not be finite) in an mset A and their multiplicities conjointly determine its *cardinality* denoted by # A or C(A) or |A|.
The set of all distinct elements of an mset A is called its *root* or *support* denoted by A^*. $\left|A^*\right|$ is called the dimension of A. A unique mset that does not contain any element is called *empty* mset, denoted by ϕ or { } or [ ]. Two msets A and B are called *equal* , denoted by $A = B$, if $m_A(x) = m_B(x)$ for all objects x. An mset A is called a *submultiset* (submset, for short) or a multisubset (msubset, for short) of an mset B, denoted as $A \subseteq B$, if $m_A(x) \le m_B(x)$ for all objects x. An mset is called the parent in relation to

its submsets. A submset of a given mset is called a *whole* if the multiplicity of each of its object is equal to its multiplicity in the parent mset, and it is called *full* if it contains all the objects of the parent mset. It should be noted here that the union of parenthood is cotextual, otherwise an mset can be seen as a submset of infinitely many msets. An mset is called finite if its root set is finite and the multiplicity of each of its distinct elements is finite, *infinite* otherwise. A cardinality bounded mset space or universe X^n (n is a nonnegative integer) is defined as the collection of all msets whose objects are drawn from the ground set X such that no object in any $A \in X^n$ occurs more than n times.

1.2 Representation of a multuset

The use of square brackets to represent an mset has become quasi-general. For example, an mset containing one occurrence of a , two occurrences of b, and three occurrences of c is denoted by [[a, b, b, c, c, c]] or [a, b, b, c, c, c] or $[a,b,c]_{1,2,3}$ or [1/a, 2/b, 3/c] or $[a^1, b^2, c^3]$ or $[a^1 b^2 c^3]$. In fact, the last form of representation as a "string" even without using any brackets, turns out to be the most compact one, especially in computational parlance. Note that the curly brackets, instead of square brackets, are also used for convenience.

Formally, for a given ground set D and a numeric set T , a mapping α:D→T is called

$$\begin{cases} \text{a "set" if } T = \{0,1\}; \\ \text{a "multiset" if } T = N, \text{the set of all natural numbers with zero;} \\ \text{a "signed" multiset" (or hybrid/shadow set) if } T = Z, \text{the set of} \\ \text{ integers; and} \\ \text{a "fuzzy" (hazy) set if } T = [0,1] \subseteq \Re, \text{a two valued Boolean} \\ \text{algebra.} \end{cases}$$

1.3 Operations Under Multisets

Union (∪): The union of msets A and B, denoted by A∪B is the smallest mset C which contains both A and B. That is, $m_{(A\cup B)}(x) = \max\{m_A(x), m_B(x)\}$, if such a maximum exists; otherwise minimum is taken which always exists. For example, for certain infinite sets like

$$X = \{\{y\}, [y]_2, [y]_3, \ldots\},$$

the maximum multiplicity of elements of X does not exists and hence ∪X={y}, the minimum.

***Intersection* (∩):** The intersection of msets A and B, denoted by A∩B , is the largest mset C which is contained in both A and B. That is,

$$m_{(A\cap B)}(x) = \min\{m_A(x), m_B(x)\}.$$

Addition or Sum or merge(+ or ⊎ or ⊕)**:** The (arithmetic) addition of msets A and B, denoted by $A+B$, is defined by $m_{A+B}(x) = m_A(x) + m_B(x)$ Note that the aforesaid definitions of mset operations could be similarly extended to an arbitrary number of msets (see [7] and [9] for details for details).

Difference and Complementation: Let A and B be two msets, and B⊂A. The (arithmectic) difference of B from A, denoted by A\B or A - B, is the mset C ⊆ A, such that $m_C(x) = m_A(x) - m_B(x)$. In general

$m_C(x) = m_A(x) - m_{A\cap B}(x) = \max\{m_A(x) - m_B(x),\ 0\}$. It is also called the relative complement of B in A. It follows that the deletion of an element x from an mset A gives rise to a new mset $A' = A - x$ such that

$m_{A'}(x) = \max\{m_A(x) - 1,\ 0\}$. On the same lines the symmetric difference of A and B, denoted by A∆B, is the mset C such that

$m_C(x) = |m_A(x) - m_B(x)|$.

It may be observed that an element may occur in an msubset of a given mset and also in its relative complement and consequently some of the outcomes of the aforsaid definition turn out to be disturbing (see [4], [3] and [1] for further details). For example, if $A = [a,b]_{4,5}$ and $B = [a,b]_{2,3}$, then $B \subset A$ and

$A - B = [a,b]_{2,2} \subset B$, contradicting the classical laws of Crisp Set Theory.

Accordingly for a given mset $Y, \langle \ \wp(Y), \cup, \cap, -, \varphi, Y \ \rangle$ is only a lattice and not a Boolean algebra (see [4] and [10] for details). It may also be noted that the aforsaid classical law holds if the complementation operator is restricted by way of introducing a predicate set in multiset context as follows:

$$[\ \mathrm{Set}(Y) \wedge X \subseteq Y] \rightarrow [\mathrm{Set}(X) \wedge \mathrm{Set}(Y - X) \wedge X \cap (Y - X) = \phi \wedge X \cup (Y - X) = Y\ ].$$

Let us recall that the whole msubsets behave Set-like. That is, if X is a whole in Y, then $X \cap (Y - X) = \phi$ and $X \cup (Y - X) = Y$.

Cartesian or (Direct) Product: Let (m / x, n / y) / k denote that x occurs m times, y occurs n times and the ordered pair (x, y) occurs k times. Also in order to comprehend the ordered n-tuples ($m_1 / x_1, m_2 / x_2, \cdots, m_n / x_n$)/k, let $C_i(x_1, x_2, \cdots, x_i, \cdots x_n)$) denote the count of the $i\,th$ coordinate in the tuple for $i = \overline{1,n}$.

The cartesian product $A_1 \times A_2 \times \cdots \times A_n$ of nonempty msets can be defined as the mset of ordered n-tuples

$(x_1, x_2, \cdots, x_i, \cdots x_n)$ where $x_i \in P_i$ and

$m_{A_1\times A_2\times\cdots\times A_n}\ (x_1, x_2, \cdots, x_i, \cdots, x_n) = \prod P_i,\ \ i = \overline{1,n}$.

Follows that $(\times A)^n$ can be defined as

$$m\,(\times A)^n\,(x_1, x_2, \cdots, x_i, \cdots x_n) = \prod m_A(x_i),\ \ i = \overline{1, n}.$$

For example, if A = [1 / x, 2 / y] and B = [2 / x, 3 / z] then

A×B = {(1 / x, 2 / x) /2, (1/x, 3/z)/3, (2/y, 2/z)/4, (2/y, 3/z)/6}

and A×A={(1/x, 1/x)/1, (1/x, 2/y)/2, (2/y, 1/x)/2, (2/y, 2/y)/4}.

1.4 Multiset Relations

Let X^n be a cardinality-bounded mset space defined on a set X and $A \in X^n$. Any subset R of $A \times A$ is called an mset relation on A, symbolized $R : A \to A,$ where every member of R has a count $C_1(x, y)$ and $C_2(x, y)$. Also m/x is $R-related$ to n/y is symbolized as $(m/x, n/y) \in R$ or $m/x\ R\ n/y$. Formally $R = \{(m/x, n/y)/mn : (m/x, n/y) \in^{mn} R\}$.

Recall that $(m/x, n/y) \in^{mn} R$ actually means that the ordered pair (x, y) occurs mn times in R. The domain of a relation R on a set A (Dom R, for short) $= \{x \in^s A : \exists y \in^t A \text{ such that } s/x\ R\ t/y\}$,where $C_{dom}R(x) = Sup\{C_1(x, y) : x \in^s A\}$, which always exists for the cardinality bounded msets. The range of R on a set A (Ran R for short) $= \{x \in^t A : \exists x \in^s A \text{ such that } s/x\ R\ t/y\}$, where $C_{ran}R(x) = Sup\{C_2(x, y) : y \in^t A\}$. For example, let $A = [5/x, 7/y, 9/z]$ be an mset. Then,

$$R = \left\{\begin{array}{l}(2/x, 3/y)/6, (4/x, 3/z)/12, (4/y, 5/z)/20, (6/y, 5/x)/30, \\ (4/z, 4/z)/16, (5/z, 3/x)/15, (3/z, 6/y)/18\end{array}\right\}$$

is an mset relation on A with dom $R = [4/x, 6/y, 5/z]$, and ran $R = [5/x, 6/y, 5/z]$.

Reflexive mset relation: An mset relation R on a set A is called "reflexive", if $(m/x, m/x) \in R$ for every $m/x \in A.$

Symmetric mset relation: An mset relation R on a set A is called "symmetric", if $(m/x, n/y) \in R$ implies $(n/y, m/x) \in R$ for $m/x,\ n/y \in A.$

Transitive mset relation: An mset relation R on a set A is called "trasitive", if $(m/x, n/y) \in R$ and $(n/y, p/z) \in R$ implies $(m/x, p/z) \in R$ for $m/x, n/y, p/z \in A.$

Equivalence mset relation: An mset relation R on a set A is called an *"equivalence relation"*, if it is *reflexive, symmetric and transitive.*

For example, let $A=[4/x,\ 5/y,\ 6/z]$ and

$$R=\left\{\begin{array}{l}(4/x,4/x)/16,(4/x,6/z)/24,(6/z,4/x)/24,(5/y,5/y)/25,\\(6/z,6/z)/36\end{array}\right\}$$ then

R is an equivalence relation on A (see [2], [6], [7], and [8] for many other details).

1.5 Multiset Partitions

The mset of R -relatives:

Let R be an mset relation on A and let $m/x\in A.$ The mset of R-relatives (or R-msets) of an element $m/x\in A$, denoted by $R(m/x)$, is the mset of all A such that there exists some k satisfying $k/x\,R\,n/y.$ For example, let $A=[5/x,7/y,9/z]$ and

$$R=\left\{\begin{array}{l}(2/x,3/y)/6,(4/x,3/z)/12,(4/y,5/z)/20,\\(6/y,5/x)/30,(4/z,4/z)/16,(5/z,3/x)/15,(3/z,6/y)/18\end{array}\right\}$$ be an

mset relation on $A.$ Then the $dom\,R=[4/x,6/y,5/z]$ and $ran\,R=$ $[5/x,6/y,5/z]$. In this case,

$R(2/x)=R(4/x)=[3/y,3/z]$, $R(4/y)=R(6/y)=[5/z,5/z]$, and $R(3/z)=R(4/z)=R(5/z)=[3/x,\ 6/y,4/z]$. The aforesaid definition can be suitably extended to defining the mset of all $R-msets$ of a subset of a given mset as follows: Let $A_1\subseteq A.$ Then $R(A_1)$, the $R-relative\ mset$ of A_1, is the mset containing all those $n/y\in A$ which are $R-relative$ to some $R(A_1).$ Follows that $R(A_1)$ is the union of all $R(m/x)$ where $m/x\in A_1.$ In fact, $R(A_1)\subseteq ran\,R$ and equality holds if and only if both $A_1\ and\ A$ have the same root set. It is important to note here that for any msubset $A_k\ of\ A,$ the mset $R(A_k)$ is defined with respect to the relation $R\ on\ A.$ For example, let A as above and

$A_1=[2/x,4/y,5/z]\subseteq A,\ where\ A_1^{*}=A^{*}$ then

$R(A_1)=[5/x,6/y,5/z]=$ The union of all $R(m/x)$ for $m/x\in A_1=$ $R(2/x)\cup R(4/y)\cup R(5/z)=R(A)=ran\,R.$

However, if the root set of an msubset $A_k\ of\ A$ is a proper subset of the root set of A then $R(A_k)\subset ran\,R,\ but\ R(A_k)\neq ran\,R.$ For example, let $A_2 = [4/x, 6/y] \subseteq A$. Then

$R(A_2) = R(4/x) \cup R(6/y) = [3/y, 3/z] \cup [5/x, 5/z] =$

$[5/x, 3/y, 5/z] \subset ran\, R, but\, [5/x, 3/y, 5/z] \neq ran\, R$

Multiset Partitions: Let $\prod = \{A_i : i = \overline{1,k}\}$ is a partition of a nonempty mset A if and only if $A = \sum_{i=}^{k} A_i$, where $\left(\sum_{i=}^{k} A_i\right)(x) = \sum_{i=}^{k} A_i(x)$. In other words a partition $\prod$ of a nonempty mset A is the collection of nonempty msubsets of such that each element of A belongs to one of these elements of $\prod$ and $A_i \cap A_j = \phi$ for distinct elements A_i and A_j of $\prod$. Equivalently, a partition $\prod$ of a nonempty mset A can be defined as a collection of disjoint whole msubsets whose union is A. The elements of $\prod$ are called the "cell" or "blocks" of $\prod$. For example, let $A = [5/x, 7/y, 9/z]$. Then $\prod = \{[5/x, 7/y], [9/z]\}$ is a partition of A and the cells of $\prod$ are $[5/x, 7/y]$ and $[9/z]$.

It is easy to see that every partition of an mset A gives rise to an equivalence relation on A. For example, the partition $\prod$ in the aforesaid example determines the equivalence mset relation

$R = \{(5/x, 5/x), (5/x, 7/y), (7/y, 5/x), (7/y, 7/y), (9/z, 9/z)\}$.

Remark: An alternative formulation of an mset partition $\prod$ induced by an equivalence relation R on an mset A can be given as follows: Using the concept of an mset R - relatives, it is immediate to see that for every equivalent mset relation R on an mset A, if $m/x \in A$ and $n/y \in A$ then $m/x\, R\, n/y$ if and only if $R(m/x) = R(n/y)$ and hence $\prod = \{R(m/x) : x \in^m A\}$; that is, $\prod$ turns out to be a collection of all distinct $R - relative$ msets that are represented by elements of A. In the example taken above, the blocks $[5/x, 7/y]$ and $[9/z]$ of the partition $\prod$ are respectively represented as $R(5/x) = R(7/y) = [5/x, 7/y]$, and $R(9/z) = [9/z]$. Note that the aforesaid representation of the block of an mset partition is not unique.

2. Monoids of Partitions of a multiset

Let A be a cardinality-bounded nonempty mset and $\prod(A)$ be the collection of all partitions of A. In order to avoid nesting of brackets, we shall represent an mset block $\left[P_k\right]$ as $\overline{P_k}$. We define a binary operation $*$ on $\prod(A)$ as follows: Let $P=\left\{\overline{P_1},\ \overline{P_2},\cdots,\overline{P_k}\right\}$ and $Q=\left\{\overline{Q_1},\ \overline{Q_2},\ \cdots,\overline{Q_k}\right\}$ be two partitions of A. Let $P*Q$ be defined as the mset of nonempty intersections of every element of P with every elements of Q. It is easy to see that the operation * is both associative and commutative and the partition consisting of a unique single block constituted by A itself is the "identity" of the operation *. Thus, $\left\langle\prod(A),*\right\rangle$ or $\left\langle\prod(A),*,\left\{\overline{A}\right\}\right\rangle$ is a monoid. It may be observed that every element $T\in\prod(A)$ is "idempotent" with respect to the operation *, since $T*T=T$ holds. Following the nomenclature of monoid of sets, this operation * on $\prod(A)$ can be called the "product of partitions". For example, let $A=\left[5/x,7/y,9/z\right]$ be an mset and $P=\left\{\overline{5/x,7/y},\overline{9/z}\right\}$ and $Q=\left\{\overline{5/x,9/z},\overline{7/y}\right\}$ are two partitions of A. Then $P*Q=\left\{\overline{5/x},\overline{7/y},\overline{9/z}\right\}$.

Here $\prod(A)=$

$$\left\{\left\{\overline{5/x},\overline{7/y},\overline{9/z}\right\},\left\{\overline{5/x,7/y},\overline{9/z}\right\},\left\{\overline{5/x},\overline{7/y,9/z}\right\},\left\{\overline{5/x,9/z},\overline{7/y}\right\},\left\{\overline{5/x,7/y,9/z}\right\}\right\}.$$

Clearly, $\left\langle\prod(A),*\right\rangle$ is monoid and $\left\{\overline{5/x},\overline{7/y},\overline{9/z}\right\}$ is the identity element.

It is interesting to see that another operation, denoted by $\oplus$, can be defined on $\prod(A)$ such that $\left\langle\prod(A),\oplus\right\rangle$ is also a monoid. We define $\oplus$ as follows: Let A be an mset and $P=\left\{\overline{P_1},\ \overline{P_2},\cdots,\overline{P_k}\right\}$ and $Q=\left\{\overline{Q_1},\ \overline{Q_2},\ \cdots,\overline{Q_k}\right\}$ be two partitions of A. A subset T of A belongs to $P\oplus Q$ if

i) T is the union of one or more elements of P,

ii) T is the union of one or more elements of Q,

iii) No subsets of T satisfies i) and ii) except T itself.

It follows that $\oplus$ is both associative and commutative. The partition consisting of single element of A is the identity of the operation $\oplus$ on $\prod(A)$. For example, $\left\{\overline{5/x},\overline{7/y},\overline{9/z}\right\}$ is the identity with respect to $\oplus$ on $\prod(A)$ in the example taken above.

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References

[1] [Bli 89] Blizard, W. D., "Multiset Theory", Notre Dame J. of Formal Logic, 30, pp. 36-66(1989).

[2] [GJ 09] Girish, K. P. and John, S. J. "Relations and Functions in Multiset Context", Information Sciences 179 (6), pp. 758 - 768, (2009).

[3] [Hic 80] Hickman, J. L., " A Note on the Concept of Multiset " Bulletin of Australian Mathematical Society 22 pp. 211 - 217(1980).

[4] [Kun 81] Knuth, D. E, "The Art of Computer Programming", Vol. 2 Seminumerical Algorithms, Second Edition, Addison-Wesley (1981).

[5] [Koh 78] Kohavi, Z., " Switching and Finite Automata Theory", Second Edition, McGraw -Hill (1978).

[6] [Pet 76] Peterson, J. L., " Computation sequence sets" Journal of Computer System Science 13(1), pp. 1- 24, (1976).

[7] [Pet 97] Petrovsky, A. B. " Structuring Techniques in Multiset Spaces", Spring pp.174 - 180 (1997).

[8] [SISS 09] Singh, D., Ibrahim, A. M. , Salihu, A. and Singh, J. N.," Multiset Relations and Thier Matrix Representations", Jnanabha (Journal of Vijnana Parishad of India, Vol 39, pp. 1-12, (2009).

[9] [SIYS 07] Singh, D. Ibrahim, A.M., Yohanna, T. and Singh, J. N.," An Overview of Applications of Multisets", Novi Sad J. of Mathematics, Vol. 37, number 2, pp. 73 -92 (2007).

[10] [SIYS 07] Singh, D. Ibrahim, A.M., Yohanna T. and Singh J. N.," Complementation in Multiset Theory" International Math. Forum, vol. 6 (38) pp. 1877 - 1884, (2011).

[11] [TM 97] Tremblay, J.P. and Manohar, R., "Discrete Mathematical Structures with. Applications to Computer Science", Tata McGraw-Hill (1997).