

# Extending Viviani's Theorem to Special Polygons Using Sketchpad

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## Abstract

In this paper I describe a classroom experience in which a group of prospective secondary mathematics teachers used Sketchpad to discover and extend Viviani's theorem to special polygons. While Sketchpad was instrumental in discovering and visualizing Viviani's theorem and some of its extensions, the class also discussed proofs to gain further insights into understanding why the proposed theorems hold for the corresponding polygons.

## Introduction

As a long-life learner of mathematics, I experience great joy when I discover and extend mathematical problems, conjectures, and theorems on my own. As a teacher of mathematics, nothing gives me greater happiness than my students discovering a conjecture on their own. One of the most powerful tools for supporting students' explorations of, and engagement with, mathematical concepts and processes is Dynamic Geometry such as the Geometer's Sketchpad (GSP, Jackiw, 2001), GeoGebra (Hohenwarter, 2002), and Cabri (Texas Instruments, 1998).

One of the most effective features of Dynamic Geometry, and technology in general, is its capacity to allow learners to represent complex mathematical ideas in new ways and to manipulate "abstract entities in a 'hands-on' way" (Perkins, Schwartz, West, & Wiske, 1995). It also offers unparalleled opportunities to investigate mathematical problems deeply. Of course, these benefits are not realized automatically by just using Dynamic Geometry. The investigations should be supplemented with the explanatory power of proof. Learners should not only use Dynamic Geometry to discover conjectures and verify them empirically, but they should construct mathematical arguments to understand why a conjecture is true. A mathematical argument allows learners to connect concepts and representations and possibly extend the conjecture to other mathematical situations.

In this paper I relate a classroom experience in which a group of prospective secondary mathematics teachers investigated and extended Viviani's theorem to special polygons. The class started with the discovery and proof of said theorem. Needless to say, GSP facilitated its discovery.

## Discovering and Proving Viviani's theorem

I enjoy presenting mathematical problems embedded in "real-world" situations to further motivate students to explore and solve them. The story embedded in the problem also often provides us with a sense of how we can use

mathematics to model problems. I guided my students to discover Viviani's theorem using the following applied context:

Three towns are the vertices of an equilateral triangle. The sides of the triangle are the roads that connect the towns. A picnic area will be constructed such that the sum of its distances to the roads is as small as possible. 1) What are all the possible locations for the picnic area? 2) For practical reasons, what is the best location for the picnic area? Justify your response.

Before students used GSP to gain insight into the solution of the problem, I asked them to predict what the optimal place to construct the picnic area is. Most students predicted that the optimal point was the center of the triangle. Students then used GSP to construct the configuration (Fig. 1) and verify empirically their initial conjecture.

As students dragged point P to verify their prediction, they noticed an interesting and surprising pattern: the sum of the distances does not change no matter what interior point they chose. A few of them noticed that the pattern also holds when P is on any side of the triangle. After naming this conjecture in honor of its original discoverer, the class summarized the results as follows:

*Viviani's conjecture:* Let P be any point in the interior of an equilateral triangle or on any of its sides. The sum of the distances from P to the sides of the triangle is a constant.

Examining the data students concluded that, theoretically, any point in the interior of the triangle or on the sides of the triangle satisfied the condition. For practical reasons, they excluded any point on the sides of the triangle as possible locations of the picnic area and suggested the center point because it is equidistant from the roads.

After having formulated a conjecture, which seemed very plausible in light of the accurate measurements provided by GSP, students knew that a proof was needed to elevate our conjecture to the rank of a theorem. A student suggested a proof based on analytic geometry.

The proof combining geometry and algebra is one of the most straightforward proofs of Viviani's theorem because knowing basic algebraic tools allows constructing the proof almost effortlessly. The proof is as follows:

First, we place the equilateral triangle ABC as indicated in figure 2. To simplify the algebra, use  $2s$  as the length of the side. Thus, vertex B has coordinates  $B(2s, 0)$  and vertex C has coordinates  $(s, \sqrt{3}s)$ . Second, we find that the equations of the sides of the triangles are  $y = 0$  ( $\overline{AB}$ ),  $y = \sqrt{3}x$  ( $\overline{AC}$ ), and  $y = -\sqrt{3}x + 2\sqrt{3}s$  ( $\overline{BC}$ ). Using the fact that the distance from point  $P(h, k)$  to line  $Ax + By + C = 0$  is  $\frac{|Ah + Bk + C|}{\sqrt{A^2 + B^2}}$  we obtain  $PD + PE + PF =$

$$k + \frac{|-\sqrt{3}h - k + 2\sqrt{3}s|}{2} + \frac{|\sqrt{3}h - k|}{2} = k + \frac{-\sqrt{3}h - k + 2\sqrt{3}s}{2} + \frac{\sqrt{3}h - k}{2} = \sqrt{3}s.$$

Notice that the absolute values are unnecessary because P is an interior point of the triangle.

The inquisitive learner may wonder whether the fact that  $PD + PE + PF = \sqrt{3}s$  is significant and whether  $\sqrt{3}s$  has an interesting geometric interpretation.

A close examination of figure 2 reveals that  $\sqrt{3}s$  is the length of the altitude of the equilateral triangle. Notice that the analytic proof is not only a convincing argument that the sum of the distances from an interior point of an equilateral triangle to its sides is a constant but also a means to discover new relationships (de Villiers, 1990, Hanna, 2000). In this case the proof led students to discover that  $PD + PE + PF$  is the height of the equilateral triangle.

One habit of mind that we need to promote among mathematics learners, particularly prospective teachers, is the process of searching for multiple arguments to justify a claim. Each argument can shed a different insight into the connections of mathematical ideas. One of the simplest, more economical and more elegant proofs of Viviani involve the use of area:

Draw segments connecting point P to each of the vertices of the triangle. These segments partition the triangle into three smaller triangles as indicated in figure 3. Let  $s$  be the length of the side of the triangle. On one hand,  $\text{Area}(\triangle ABC) = \frac{sh}{2}$ . On the other hand,  $\text{Area}(\triangle ABC) = \text{Area}(\triangle ABP) + \text{Area}(\triangle BCP) + \text{Area}(\triangle CAP) = \frac{s \cdot PD}{2} + \frac{s \cdot PE}{2} + \frac{s \cdot PF}{2} = \frac{s}{2}(PD + PE + PF)$ .

Thus we conclude that  $\frac{s}{2}(PD + PE + PF) = \frac{sh}{2}$  or  $PD + PE + PF = h$ .

Notice that the area proof also helps us to discover the fact that the sum of distances from an interior point of an equilateral triangle to its sides is the measure of its altitude. Both the analytic geometry and area proof fulfill the functions of proof as justification, conviction, and discovery. However, the argument involving area has more explanatory power. But there are other reasons why learners should see more than one proof of a theorem. Winicki-Landman (1998) argues that students should be exposed to different types of proofs so they learn to appreciate the beauty and elegance of proof by comparing and contrasting different proofs of the same theorem.

To model the process of extending mathematical problems, conjectures, and theorems, I asked students whether we could extend Viviani's theorem to other geometric figures. Some suggested considering quadrilaterals and polygons. We first considered quadrilaterals. Since students wanted to use GSP immediately, I asked them to think for what quadrilaterals Viviani's theorem may hold without using the software. Of course, students realized that Viviani's theorem does not hold for general quadrilaterals so we started with the most special of the common quadrilaterals: a square. A simple drawing was enough to discover and understand that Viviani's theorem holds for squares (Fig. 4). We formulated our theorem as follows:

Let P be an interior point of a square. The sum of the distances from P to its sides is the semi-perimeter of the square.

We next considered a rectangle and discovered a similar relationship. The pattern, however, had to be modified for a rhombus: The sum of the distances from an interior point P of a rhombus to its sides is the sum of the distances between the parallel sides (Fig. 5). A similar result holds for a parallelogram. Notice that in both cases the class reinterpreted the geometric meaning of the constant  $PE + PF + PG + PH$  since it is not the semi-perimeter of the rhombus.

Our next task was to investigate whether Viviani's theorem could be extended to other quadrilaterals. After an unsuccessful search in the realm of quadrilaterals, the class turned to regular polygons. We first considered a regular pentagon (Fig. 6). As students dragged point P in the interior of the pentagon, they confirmed experimentally that the sum of the distances of an interior point of a regular pentagon to its sides is a constant.

Our next task was to provide a proof and discover the geometrical meaning of the constant. As we did for equilateral triangles, we constructed segments from P to each of the vertices of polygon ABCDE to represent its area in two different ways. On one hand,  $\text{Area}(\text{ABCDE}) = \text{Area}(\triangle \text{ABP}) + \text{Area}(\triangle \text{BCP}) + \text{Area}(\triangle \text{CDP}) + \text{Area}(\triangle \text{DEP}) + \text{Area}(\triangle \text{EAP}) = \frac{s}{2} (PF + PG + PH + PI + PJ)$ .

On the other hand,  $\text{Area}(\text{ABCDE}) = \frac{5sa}{2}$ , where  $a$  is the apothem of the regular pentagon. Therefore, we conclude that  $PF + PG + PH + PI + PJ = 5a$ .

As hindsight, a student suggested dragging point P to the center of the regular pentagon to visualize that  $PF + PG + PH + PI + PJ$  equals five times the apothem of the regular pentagon (Fig. 8).

The class then extended and proved the result for a regular  $n$ -gon: Let P be an interior point of a regular  $n$ -gon. The sum of the distances from P to the sides of the polygon (S) is a constant and is equal to  $na$ , where  $a$  is the apothem of the polygon.

Previously we had defined the altitude of a regular polygon with an even number of sides as the distance between any pair of parallel sides. For a regular polygon with an odd number of sides, we defined its altitude as the distance from a vertex to the opposite side. I asked students whether we could represent S in terms of  $h$ , the altitude. A student quickly replied that if the polygon has an even number of sides, then the sum equals to  $\frac{nh}{2}$ . In spite of our efforts, we were not able to find an expression for S in terms of  $h$  for regular polygons with an odd number of sides.

At this point the class thought that we had finished extending Viviani's theorem to polygons. The students then were surprised when I asked them whether Viviani's theorem could be extended to other non-regular polygons. The class previously had defined and explored parallelogons (polygons whose opposite sides are parallel), equilateral polygons, and equiangular polygons so it was natural for some students to consider those polygons as potential situations to which extend Viviani's theorem. The class first considered a parallelo-hexagon (Fig. 9).

All students were able to see that the sum of the distances from an interior point of a parallelo-hexagon to its sides was a constant: the sum of the distances between opposite parallel sides, so no GSP measurements were needed to verify their claim. The class then extended this result to parallelo- $2n$ -gons. The class next considered an equilateral pentagon (Fig. 10).

Students were surprised that Viviani's theorem could be extended to equilateral pentagons. To explain why this was the case, one student provided the following incomplete argument:  $\text{Area}(\text{ABCDE}) = \text{Area}(\triangle \text{ABP}) + \text{Area}(\triangle \text{BCP}) + \text{Area}(\triangle \text{CDP}) + \text{Area}(\triangle \text{DEP}) + \text{Area}(\triangle \text{EAP}) =$

$\frac{s}{2}(PF + PG + PH + PI + PJ) = \frac{s}{2}S$ , where  $s$  is the length of the side of the equilateral pentagon. The student then said that she was “stuck” because she wanted to express Area(ABCDE) in terms of the apothem. I asked the student to solve for  $S$  in terms of the area of ABCDE ( $S = \frac{2 \times \text{Area}(ABCDE)}{s}$ ) but she did not see that this expression meant that  $S$  was a constant. Another student said “Oh, since Area(ABCDE) and  $s$  are constants, we can conclude that  $\frac{2 \times \text{Area}(ABCDE)}{s}$  is also a constant.” Next, the class considered an equiangular hexagon (Fig. 11).

The students realized that an equiangular hexagon was also a parallelogram and, thus, Viviani’s theorem held for an equiangular hexagon. The class then proposed to investigate the case of an equiangular pentagon.

Before constructing the GSP configuration, I asked students to predict whether Viviani’s theorem held for equiangular polygons. Since an equiangular polygon with an odd number of sides does not necessarily have congruent sides or parallel sides, some student predicted that Viviani’s theorem did not hold for equiangular polygons with an odd number of sides. They were then surprised to discover that their prediction was incorrect (Fig. 12).

The class then took the challenge to explain logically the discovery but without success. I suggested to the students to embed the equiangular pentagon into a regular pentagon as shown in figure 13.

A student then visualized the proof immediately and provided an argument along the following lines:

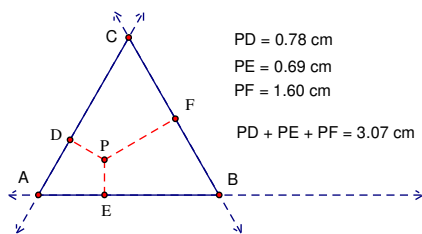
$PF + PG + PH + PI + PJ = (PF + PO + PQ + PR + PJ) - (GO + HQ + IR) = k$ , for  $k$  a real number, since  $PF + PO + PQ + PR + PJ$  and  $GO + HQ + IR$  are constants, and the difference of two constants is a constant.  $PF + PO + PQ + PR + PJ$  is a constant because AKLMN is a regular pentagon.  $GO + HQ + IR$  is a constant because is the sum of the distance between parallel sides of the pentagons. The corresponding sides of the two pentagons are parallel because their corresponding angles are congruent.

The class thought that this proof was also simple, economical, and elegant. The class continued their Viviani’s adventures in other directions but that is another story that should be told another time.

### Conclusion

Viviani’s theorem is one of my favorite theorems because it can be proved using different strategies and can be extended to a variety of geometric figures. I present it to my students every time I teach the geometry course for prospective secondary mathematics teachers. Most of them come to appreciate its beauty, elegance, and simplicity. Although our extensions are not original, all of us, the instructor and the students, experience the thrill of formulating and discovering new theorems, at least new to us.

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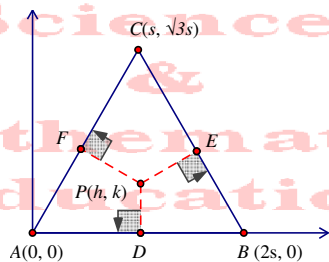


Figure 2: Diagram for the Analytic proof of Viviani's theorem

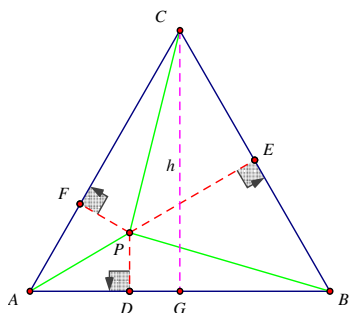


Figure 3:  $\text{Area}(\triangle ABC) = \text{Area}(\triangle ABP) + \text{Area}(\triangle BCP) + \text{Area}(\triangle CAP)$

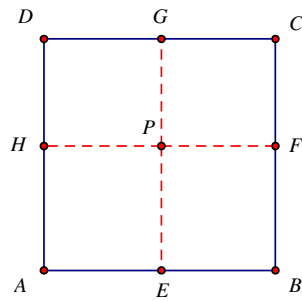


Figure 4: Viviani's theorem holds for squares

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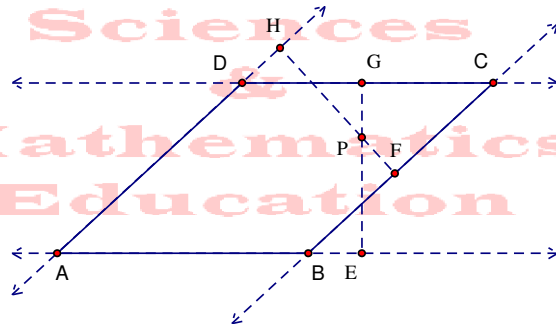


Figure 5:  $PE + PF + PG + PH$  is a constant for rhombi and parallelograms

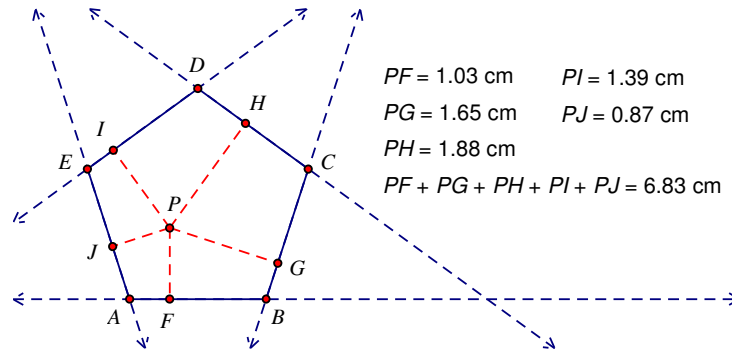


Figure 6:  $PF + PG + PH + PI + PJ$  is a constant for an interior point P of a regular pentagon

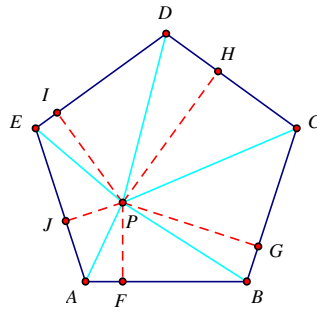


Figure 7:  $\text{Area}(ABCDE) = \text{Area}(\triangle ABP) + \text{Area}(\triangle BCP) + \text{Area}(\triangle CDP) + \text{Area}(\triangle DEP) + \text{Area}(\triangle EAP)$

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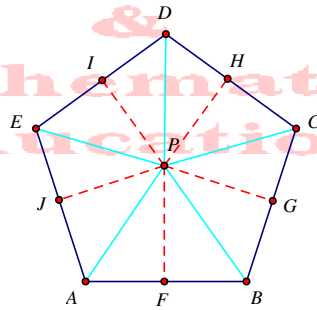


Figure 8: Visualizing that  $PF + PG + PH + PI + PJ = 5a$

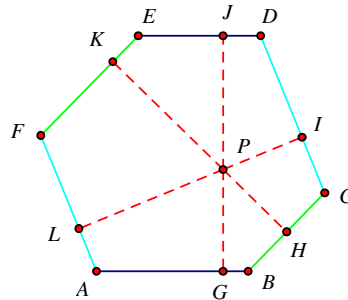


Figure 9: S is constant for a parallelo-hexagon when P is an interior point



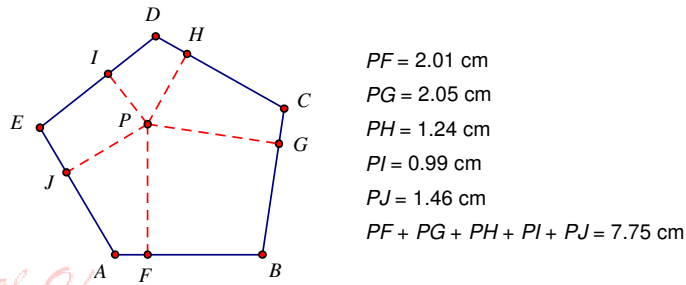


Figure 10: S is constant for an equilateral pentagon when P is an interior point

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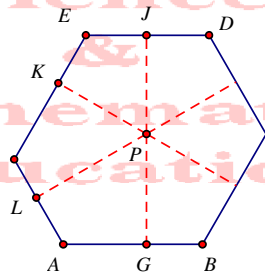


Figure 11: S is constant when P is an interior point of equiangular hexagon

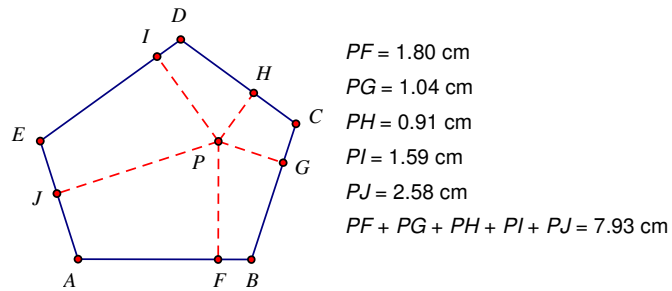


Figure 12: S is constant for an interior point P of an equiangular pentagon

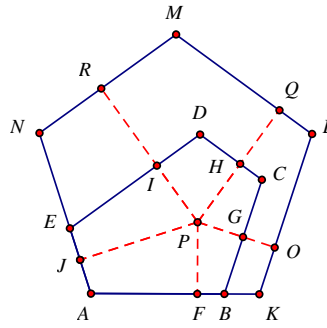


Figure 13: Every equiangular pentagon can be embedded into a regular pentagon

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