A converse of the mean value theorem for integrals of functions of one or more variables

Diane Denny, Ph.D. †

Abstract

Let f be a continuous function of **x** on Ω, where $\Omega \subset R^N$, N ≥ 1, is a bounded, open, convex, connected set. We prove that if $f(\vec{c})$ is not the absolute maximum or absolute minimum value of f in Ω, where $\vec{c} \in \Omega$ is a given point, then there exists a set $S \subset \Omega$ such that $f(\vec{c}) = \frac{1}{|S|}\int_S f(\vec{x})d\vec{x}$.

Introduction

One version of the Mean Value Theorem of integral calculus states that if f is a continuous function of $\vec{x}$ on a given compact, connected set $V \subset R^N$, then there exists a point $\vec{c} \in V$ such that $\frac{1}{|V|}\int_V f(\vec{x})d\vec{x} = f(\vec{c})$ (see, e.g., [2]).

The question to be considered here is: If $f(\vec{c})$ is the value of a continuous function f at a given point $\vec{c} \in \Omega$, where $\Omega \subset R^N$, N ≥ 1, is a bounded, open, convex, connected set, then does there exist a set $S \subset \Omega$ such that

$$f(\vec{c}) = \frac{1}{|S|}\int_S f(\vec{x})d\vec{x}\ ?$$

In this paper, we prove that if $f(\vec{c})$ is not the absolute maximum or absolute minimum value of f in Ω, then there exists a set $S \subset \Omega$ such that

$$f(\vec{c}) = \frac{1}{|S|}\int_S f(\vec{x})d\vec{x}.$$

In previous related work by other researchers, several papers have studied the converse of the Mean Value Theorem for functions of one variable. Tong and Braza [4] proved that given a continuous function $f:[a,b] \to R$ and given $c \in (a,b)$ such that c is not an accumulation point of the set $\{x \in (a,b): f(x) = f(c)\}$ and c is not a local extremum point of f, then there exists $(\alpha,\beta) \subset (a,b)$, where $c \in (\alpha,\beta)$, such that $\int_\alpha^\beta f(x)dx = f(c)(\beta-\alpha)$.

In related work on the Mean Value Theorem for differentiable functions F of one variable, Tong and Braza [5] and Mortici [3] proved that if F is continuous on $[a,b]$ and differentiable on (a,b), then there exists an interval $(\alpha,\beta)\subset(a,b)$ such that $F(\beta)-F(\alpha)=F'(c)(\beta-\alpha)$, provided F' satisfies certain hypotheses. These hypotheses are that either $F'(c)$ is not a local extremum value of $F'(x)$ on (a,b) and c is not an accumulation point of the set $\{x\in(a,b): F'(x)=F'(c)\}$, in which case $c\in(\alpha,\beta)$, or alternatively that $F'(c)$ is not a global extremum value of $F'(x)$ on (a,b), in which case c is not necessarily inside (α,β). Almeida [1] proved that if F is continuous on $[a,b]$ and differentiable on (a,b), then there exists an interval $(\alpha,\beta)\subset(a,b)$ with $c\in[\alpha,\beta]$ such that $F(\beta)-F(\alpha)=F'(c)(\beta-\alpha)$, provided that there exists $k_0>0$ such that $(c-k_0,c+k_0)\subset(a,b)$ and $F'(c-k)\le F'(c)\le F'(c+k)$ for all $k\in(0,k_0)$.

We have not seen work related to the converse of the Mean Value Theorem for integrals of functions of several variables.

A converse of the mean value theorem for integrals

We present the results of this paper in two theorems. The first theorem considers the existence of a set $S\subset\Omega$ such that $f(\vec{c})=\dfrac{1}{|S|}\int_S f(\vec{x})d\vec{x}$.

The second theorem concerns conditions under which $\vec{c}\in S$ for the special case in which $S=[a_1,b_1]\times[a_2,b_2]\times\ldots\times[a_N,b_N]$.

We begin by proving the following theorem:

Theorem 1: Let $f:\Omega\to R$ be a continuous function of $\vec{x}\in\Omega$, where $\Omega\subset R^N$, $N\ge 1$, is a bounded, open, convex, connected set. Let $f(\vec{c})$ be the value of f at a given point $\vec{c}\in\Omega$.
If there exists an open set $A\subset\Omega$ such that $f(\vec{x})=f(\vec{c})$ for all $\vec{x}\in A$, then $f(\vec{c})=|A|^{-1}\int_A f(\vec{x})d\vec{x}$.
If there does not exist an open set $A\subset\Omega$ such that $f(\vec{x})=f(\vec{c})$ for all $\vec{x}\in A$, then we have the following cases:

Case 1: If $f(\vec{c})$ is not the absolute maximum or absolute minimum value of f in Ω, then there exists a set $S_0 \subset \Omega$ such that

$$f(\vec{c}) = |S_0|^{-1} \int_{S_0} f(\vec{x}) d\vec{x}.$$

Case 2: If $f(\vec{c})$ is the absolute maximum value of f in Ω, then there exists a set $S_1 \subset \Omega$ and a positive constant ε_1 such that

$$f(\vec{c}) = |S_1|^{-1} \int_{S_1} f(\vec{x}) + \varepsilon_1 d\vec{x}.$$

Case 3: If $f(\vec{c})$ is the absolute minimum value of f in Ω, then there exists a set $S_2 \subset \Omega$ and a positive constant ε_2 such that

$$f(\vec{c}) = |S_2|^{-1} \int_{S_2} f(\vec{x}) - \varepsilon_2 d\vec{x}.$$

Proof:

If there exists an open set $A \subset \Omega$ such that $f(\vec{x}) = f(\vec{c})$ for all $\vec{x} \in A$, it immediately follows that $f(\vec{c}) = |A|^{-1} \int_A f(\vec{x}) d\vec{x}$. Therefore, now suppose that there does not exist an open set $A \subset \Omega$ such that $f(\vec{x}) = f(\vec{c})$ for all $\vec{x} \in A$.

We have three possible cases to consider:

(1) Case 1 is the case in which $f(\vec{c})$ is not the absolute maximum or absolute minimum value of f in Ω.
(2) Case 2 is the case in which $f(\vec{c})$ is the absolute maximum value of f in Ω.
(3) Case 3 is the case in which $f(\vec{c})$ is the absolute minimum value of f in Ω.

Note that we are not assuming that f has an absolute maximum value or absolute minimum value in Ω.

We now consider each case separately.

Case 1: Suppose that $f(\vec{c})$ is not the absolute (global) maximum or absolute (global) minimum value of f in Ω. Let $g(\vec{x}) = f(\vec{x}) - f(\vec{c})$. Then

$g(\vec{c}) = 0$. Since $f(\vec{c})$ is not the absolute maximum value of f in Ω, there exists a point $\vec{x}_1 \in \Omega$ such that $g(\vec{x}_1) > 0$. Since $g(\vec{x})$ is continuous, there exists an open ball $B_1 = B(\vec{x}_1, \partial_1)$ of radius ∂_1 about the point $\vec{x}_1$, such that $\overline{B}_1 \subset \Omega$ and such that $g(\vec{x}) > 0$ for $\vec{x} \in B_1$.

Since $f(\vec{c})$ is not the absolute minimum value of f in Ω, there exists a point $\vec{x}_2 \in \Omega$ such that $g(\vec{x}_2) < 0$. Since $g(\vec{x})$ is continuous, there exists an open ball $B_2 = B(\vec{x}_2, \partial_2)$ of radius ∂_2 about the point $\vec{x}_2$, such that $\overline{B}_2 \subset \Omega$ and such that $g(\vec{x}) < 0$ for $\vec{x} \in B_2$.

Since $\overline{B}_1 \subset \Omega$ and $\overline{B}_2 \subset \Omega$, and since Ω is a connected open set in R^N, it follows that there exists a connected open set $U \subset \Omega$ such that $B_1 \subset U$, and such that $B_2 \subset U$, and such that the distance ∂_3 from the boundary of U to the boundary of Ω is positive, so that $\overline{U} \subset \Omega$. Therefore $B_3 = B(\vec{x}, \partial_3) \subset \Omega$ for any $\vec{x} \in U$. Let $\partial_4 = \min\{\partial_1, \partial_2, \partial_3\}$. We now define

$$G(\vec{x}) = \frac{1}{|B(\vec{x}, \partial_4)|} \int_{B(\vec{x}, \partial_4)} g(\vec{y}) d\vec{y}, \quad \text{where } \vec{x} \in U.$$

It follows that G is a continuous function of $\vec{x}$ on U, and $G(\vec{x}_2) < 0$ and $G(\vec{x}_1) > 0$, since $g(\vec{y}) < 0$ for $\vec{y} \in B(\vec{x}_2, \partial_4) \subset B_2$ and $g(\vec{y}) > 0$ for $\vec{y} \in B(\vec{x}_1, \partial_4) \subset B_1$.

Since G is continuous on the connected set U, and $G(\vec{x}_2) < 0$ and $G(\vec{x}_1) > 0$, where $\vec{x}_1 \in U$ and where $\vec{x}_2 \in U$, then by the Intermediate Value Theorem (see, e.g., [2]) there exists a point $\vec{x}_3 \in U$ such that $G(\vec{x}_3) = 0$. Therefore

$$0 = G(\vec{x}_3) = \frac{1}{|B(\vec{x}_3, \partial_4)|} \int_{B(\vec{x}_3, \partial_4)} g(\vec{x}) d\vec{x} = \frac{1}{|B(\vec{x}_3, \partial_4)|} \int_{B(\vec{x}_3, \partial_4)} f(\vec{x}) - f(\vec{c}) d\vec{x}.$$

Re-arranging terms yields

$$f(\vec{c}) = \frac{1}{|B(\vec{x}_3, \partial_4)|} \int_{B(\vec{x}_3, \partial_4)} f(\vec{x}) d\vec{x}.$$

We define $S_0 = B(\vec{x}_3, \partial_4)$ and the proof for Case 1 is complete.

Case 2: Suppose that $f(\vec{c})$ is the absolute maximum value of f in Ω. Let $B_0 = B(\vec{c}, \partial_0) \subset \Omega$ be the open ball of radius ∂_0 about the point $\vec{c}$ such that ∂_0 is the distance from $\vec{c}$ to the boundary of Ω. Let $B_1 = B(\vec{c}, \partial_1)$ be the open ball of radius $\partial_1 < 1/2\ \partial_0$ about the point $\vec{c}$. Note that $\overline{B}_1 \subset \Omega$.

Let $g(\vec{x}) = f(\vec{x}) - f(\vec{c})$. Then $g(\vec{c}) = 0$, and $g(\vec{x}) \leq 0$ for $\vec{x} \in \Omega$. Let $\varepsilon_1 = -\varepsilon_0 \min_{\vec{x}\in\overline{B}_1} g(\vec{x})$, where $0<\varepsilon_0<1$. Note that $\varepsilon_1 > 0$ (since otherwise it would follow that $\min_{\vec{x}\in\overline{B}_1} g(\vec{x}) = 0 = \max_{\vec{x}\in\overline{B}_1} g(\vec{x}) = g(\vec{c})$, which implies that $g(\vec{x}) = 0$ in $\overline{B}_1$ and so $f(\vec{x}) = f(\vec{c})$ on the set $A = B_1$, but this contradicts the assumption made at the start of the proof of this theorem that such an open set A does not exist). Also note that ε_1 can be arbitrarily small since ε_0 can be arbitrarily small. And since g is continuous on $\overline{B}_1$, it follows that there exists a point $\vec{x}_1 \in \overline{B}_1$ such that $g(\vec{x}_1) = \min_{\vec{x}\in\overline{B}_1} g(\vec{x}) = -\varepsilon_1/\varepsilon_0$.

We have $-\varepsilon_1/\varepsilon_0 = \min_{\vec{x}\in\overline{B}_1} g(\vec{x}) = g(\vec{x}_1) \leq g(\vec{x}) \leq g(\vec{c}) = \max_{\vec{x}\in\overline{B}_1} g(\vec{x}) = 0$ for $\vec{x} \in \overline{B}_1$.

Now let $h(\vec{x}) = g(\vec{x}) + \varepsilon_1$. It follows that $(1 - 1/\varepsilon_0)\varepsilon_1 = \min_{\vec{x}\in\overline{B}_1} h(\vec{x}) = h(\vec{x}_1) \leq h(\vec{x}) \leq h(\vec{c}) = \max_{\vec{x}\in\overline{B}_1} h(\vec{x}) = \varepsilon_1$ for $\vec{x} \in \overline{B}_1$. And $(1 - 1/\varepsilon_0)\varepsilon_1 < 0$, since $0<\varepsilon_0<1$ and $\varepsilon_1 > 0$.

Since h is continuous on Ω, and since $h(\vec{x}_1) < 0$ and $h(\vec{c}) > 0$, where $\vec{x}_1 \in \overline{B}_1$ and $\vec{c} \in \overline{B}_1$, it follows that exists a radius $\partial_2 < \partial_1$ such that $h(\vec{x}) < 0$ for $\vec{x} \in B(\vec{x}_1, \partial_2)$, and such that $h(\vec{x}) > 0$ for $\vec{x} \in B(\vec{c}, \partial_2)$.

We now define $H(\vec{x}) = \dfrac{1}{|B(\vec{x}, \partial_2)|} \displaystyle\int_{B(\vec{x},\partial_2)} h(\vec{y})d\vec{y}$, where $\vec{x} \in \overline{B}_1$. Note that

$B(\vec{x}, \partial_2) \subset B(\vec{c}, \partial_0) \subset \Omega$ for $\vec{x} \in \overline{B}_1$.

It follows that H is a continuous function of $\vec{x}$ on $\overline{B}_1$, and $H(\vec{x}_1) < 0$ and $H(\vec{c}) > 0$, since $h(\vec{y}) < 0$ for $\vec{y} \in B(\vec{x}_1, \partial_2)$ and $h(\vec{y}) > 0$ for $\vec{y} \in B(\vec{c}, \partial_2)$.

Since H is a continuous function of $\vec{x}$ on the connected set $\overline{B}_1$, and $H(\vec{x}_1) < 0$ and $H(\vec{c}) > 0$, where $\vec{x}_1 \in \overline{B}_1$ and where $\vec{c} \in \overline{B}_1$, then by the Intermediate Value Theorem there exists a point $\vec{x}_2 \in \overline{B}_1$ such that $H(\vec{x}_2) = 0$. Therefore

$$0 = \frac{1}{|B(\vec{x}_2, \partial_2)|} \int_{B(\vec{x}_2, \partial_2)} h(\vec{x}) d\vec{x} = \frac{1}{|B(\vec{x}_2, \partial_2)|} \int_{B(\vec{x}_2, \partial_2)} f(\vec{x}) - f(\vec{c}) + \varepsilon_1 d\vec{x} .$$

Re-arranging terms yields

$$f(\vec{c}) = \frac{1}{|B(\vec{x}_2, \partial_2)|} \int_{B(\vec{x}_2, \partial_2)} f(\vec{x}) + \varepsilon_1 d\vec{x} .$$

We define $S_1 = B(\vec{x}_2, \partial_2)$ and the proof for Case 2 is complete.

Case 3:

Suppose that $f(\vec{c})$ is the absolute minimum value of f in Ω. Then $f(\vec{x}) - f(\vec{c}) \geq 0$ for $\vec{x} \in \Omega$. Let $v(\vec{x}) = -f(\vec{x})$. And so $v(\vec{x}) - v(\vec{c}) \leq 0$ for $\vec{x} \in \Omega$, and $v(\vec{c})$ is the absolute maximum value of v in Ω. From the proof of Case 2, it follows that there exists a point $\vec{x}_3 \in \Omega$, and a radius ∂_3, and a positive constant ε_2 such that

$$v(\vec{c}) = \frac{1}{|B(\vec{x}_3, \partial_3)|} \int_{B(\vec{x}_3, \partial_3)} v(\vec{x}) + \varepsilon_2 d\vec{x} .$$

Since $v(\vec{x}) = -f(\vec{x})$, multiplying this equation by -1 yields

$$f(\vec{c}) = \frac{1}{|B(\vec{x}_3, \partial_3)|} \int_{B(\vec{x}_3, \partial_3)} f(\vec{x}) - \varepsilon_2 d\vec{x} .$$

We define $S_2 = B(\vec{x}_3, \partial_3)$ and the proof for Case 3 is complete.

This completes the proof of Theorem 1.

We now prove the following theorem:

Theorem 2: Let $f : \Omega \to R$ be a continuous function of $\vec{x} \in \Omega$, where $\Omega \subset R^N$, N $\geq$ 1, is a bounded, open, convex, connected set. Let $f(\vec{c})$ be the value of f at a given point $\vec{c} \in \Omega$.

If there exists an open set $A \subset \Omega$ such that $f(\vec{x}) = f(\vec{c})$ for all $\vec{x} \in A$, where $\vec{c} \in A$, then $f(\vec{c}) = \frac{1}{|A|}\int_A f(\vec{x})d\vec{x}$.

If there does not exist an open set $A \subset \Omega$ such that $f(\vec{x}) = f(\vec{c})$ for all $\vec{x} \in A$, where $\vec{c} \in A$, then we have the following cases:

Case 1: Suppose the spatial dimension N=1.

If $f(c)$ is not the absolute maximum or absolute minimum value of f in Ω, then there exist a_1, b_1 in Ω such that $a_1 < c < b_1$ and $f(c) = \frac{1}{b_1 - a_1}\int_{a_1}^{b_1} f(x)dx$ if and only if there exist y_1, z_1 in Ω such that $y_1 < c < z_1$ and $G(y_1), G(z_1)$ have the same sign or $G(y_1) = G(z_1) = 0$, where $G(t) = \int_c^t f(x) - f(c)dx$.

Case 2: Suppose the spatial dimension N$\geq$ 2.

If $f(\vec{c})$ is not the absolute maximum or absolute minimum value of f in Ω, then there exists a set $S \subset \Omega$, where $\vec{c} = (c_1, c_2, ..., c_N) \in S$, and where $S = [a_1, b_1] \times [a_2, b_2] \times ... \times [a_N, b_N]$, such that $f(\vec{c}) = \frac{1}{|S|}\int_S f(\vec{x})d\vec{x}$ if for j=1,2,...,N, there exist y_j, z_j such that $y_j < c_j < z_j$ and $G_j(y_j), G_j(z_j)$ have the same sign or $G_j(y_j) = G_j(z_j) = 0$, where

$$G_j(t)=\int_{c_j}^{t}\int_{a_{j-1}}^{b_{j-1}}\int_{a_{j-2}}^{b_{j-2}}\cdots\int_{a_1}^{b_1} f(x_1,x_2,...,x_j,c_{j+1},...c_N)-f(\vec{c})dx_1dx_2...dx_j$$

where a_i, b_i are determined iteratively for each i.

Proof of Theorem 2:

If there exists an open set $A\subset\Omega$ such that $f(\vec{x})=f(\vec{c})$ for all $\vec{x}\in A$, where $\vec{c}\in A$, it immediately follows that $f(\vec{c})=\frac{1}{|A|}\int_A f(\vec{x})d\vec{x}$.

Therefore, now suppose that there does not exist an open set $A\subset\Omega$ such that $f(\vec{x})=f(\vec{c})$ for all $\vec{x}\in A$, where $\vec{c}\in A$.

We consider the cases in which the spatial dimension N=1 and in which N$\geq$2 separately.

Case 1: First let N=1.

Suppose that $f(c)$ is not the absolute maximum or absolute minimum value of f in Ω. Let $g(x)=f(x)-f(c)$. Then $g(c)=0$, and $g(c)$ is not the absolute maximum or absolute minimum value of g in Ω. We define $G(t)=\int_c^t g(x)dx=\int_c^t f(x)-f(c)dx$, where $t\in\Omega=(a,b)$. Note that $G(c)=0$.

We begin by proving that there exist y_1, z_1 in Ω such that $y_1<c<z_1$ and $G(y_1)$, $G(z_1)$ have the same sign (i.e., both are positive or both are negative numbers) or $G(y_1)=G(z_1)=0$ if and only if there exist a_1, b_1 in Ω such that $a_1<c<b_1$ and $G(a_1)=G(b_1)$.

Therefore, suppose that there exist y_1, z_1 in Ω such that $y_1<c<z_1$ and $G(y_1)$, $G(z_1)$ have the same sign or $G(y_1)=G(z_1)=0$. If $G(y_1)=G(z_1)=0$ then we are done. The desired result that $G(a_1)=G(b_1)$ holds with $a_1=y_1$ and $b_1=z_1$.

Next, suppose that $G(y_1)$, $G(z_1)$ have the same sign. If $G(y_1) = G(z_1)$ then we are done. The desired result that $G(a_1) = G(b_1)$ holds with $a_1 = y_1$ and $b_1 = z_1$.

Next suppose that $G(y_1)$, $G(z_1)$ have the same sign and $G(y_1) \neq G(z_1)$. First, assume that $0 < G(y_1) < G(z_1)$. Recall that $G(c) = 0$ and that $y_1 < c < z_1$. Therefore, by the continuity of $G(t)$ and the Intermediate Value Theorem, it follows that there exists z_2 in Ω such that $c < z_2 < z_1$ and $G(z_2) = G(y_1)$. The desired result that $G(a_1) = G(b_1)$ holds with $a_1 = y_1$ and $b_1 = z_2$.

Similarly, if $0 < G(z_1) < G(y_1)$, it follows that there exists y_2 in Ω such that $y_1 < y_2 < c$ and $G(y_2) = G(z_1)$. The desired result that $G(a_1) = G(b_1)$ holds with $a_1 = y_2$ and $b_1 = z_1$.

And if $G(z_1) < G(y_1) < 0$, it follows that there exists z_3 in Ω such that $c < z_3 < z_1$ and $G(z_3) = G(y_1)$. The desired result that $G(a_1) = G(b_1)$ holds with $a_1 = y_1$ and $b_1 = z_3$.

Finally, if that $G(y_1) < G(z_1) < 0$, it follows that there exists y_3 in Ω such that $y_1 < y_3 < c$ and $G(y_3) = G(z_1)$. The desired result that $G(a_1) = G(b_1)$ holds with $a_1 = y_3$ and $b_1 = z_1$.

Conversely, suppose that there exist a_1, b_1 in Ω such that $a_1 < c < b_1$ and $G(a_1) = G(b_1)$ .Then $G(a_1) = G(b_1) = 0$ or $G(a_1) = G(b_1) \neq 0$, in which case $G(a_1), G(b_1)$ have the same sign. Therefore, there exist y_1, z_1 in Ω such that $y_1 < c < z_1$ and $G(y_1)$, $G(z_1)$ have the same sign or $G(y_1) = G(z_1) = 0$, where we define $y_1 = a_1$ and $z_1 = b_1$.

Therefore, there exist y_1, z_1 in Ω such that $y_1 < c < z_1$ and $G(y_1)$, $G(z_1)$ have the same sign or $G(y_1) = G(z_1) = 0$ if and only if there exist a_1, b_1 in Ω such that $a_1 < c < b_1$ and $G(a_1) = G(b_1)$.

Since $G(t)=\int_c^t g(x)dx=\int_c^t f(x)-f(c)dx$, it immediately follows that

$$G(a_1)=G(b_1) \text{ if and only if } f(c)=\frac{1}{b_1-a_1}\int_{a_1}^{b_1} f(x)dx.$$

Therefore, there exist y_1, z_1 in Ω such that $y_1<c<z_1$ and $G(y_1)$, $G(z_1)$ have the same sign or $G(y_1)=G(z_1)=0$ if and only if there exist a_1, b_1 in Ω such that $a_1<c<b_1$ and $f(c)=\frac{1}{b_1-a_1}\int_{a_1}^{b_1} f(x)dx$.

This completes the proof of Case 1 of the theorem.

Case 2: Next, suppose N ≥ 2.

Suppose that $f(\vec{c})$ is not the absolute maximum or absolute minimum value of f in Ω. Let $g(\vec{x})=f(\vec{x})-f(\vec{c})$. Then $g(\vec{c})=0$, and $g(\vec{c})$ is not the absolute maximum or absolute minimum value of g in Ω.

We next prove there exists a set $S\subset\Omega$, where $\vec{c}=(c_1,c_2,\ldots,c_N)\in S$ and where $S=[a_1,b_1]\times[a_2,b_2]\times\ldots\times[a_N,b_N]$, such that

$f(\vec{c})=\frac{1}{|S|}\int_S f(\vec{x})d\vec{x}$ if for j=1,2,...,N, there exist y_j, z_j such that $y_j<c_j<z_j$ and $G_j(y_j), G_j(z_j)$ have the same sign or $G_j(y_j)=G_j(z_j)=0$, where

$$G_j(t)=\int_{c_j}^{t}\int_{a_{j-1}}^{b_{j-1}}\int_{a_{j-2}}^{b_{j-2}}\ldots\int_{a_1}^{b_1} g(x_1,x_2,\ldots,x_j,c_{j+1},\ldots c_N)dx_1dx_2\ldots dx_j.$$

To prove this result, we will repeatedly apply the proof used in Case 1 for N=1.

We begin by defining $G_1(t)=\int_{c_1}^{t} g(x_1,c_2,\ldots c_N)dx_1$

for t such that $(t,c_2,\ldots,c_N)\in\Omega$. Recall that $(c_1,c_2,\ldots,c_N)\in\Omega$. Also, recall that Ω is a bounded, open, convex, connected set, so that $(x_1,c_2,\ldots,c_N)\in\Omega$ on the interval of integration. And $G_1(c_1)=0$.

By the proof from Case 1 for N=1, if there exist y_1, z_1 such that $y_1 < c_1 < z_1$ and $G_1(y_1)$, $G_1(z_1)$ have the same sign or $G_1(y_1) = G_1(z_1) = 0$, then there exist a_1, b_1 such that $a_1 < c_1 < b_1$ and $G_1(a_1) = G_1(b_1)$. It immediately follows that $\int_{a_1}^{b_1} g(x_1, c_2, \ldots c_N)\,dx_1 = 0$.

Next, we define $G_2(t) = \int_{c_2}^{t} \int_{a_1}^{b_1} g(x_1, x_2, c_3, \ldots c_N)\,dx_1 dx_2$ for t such that $(x_1, t, c_3, \ldots, c_N) \in \Omega$ for $a_1 \le x_1 \le b_1$. Note that $(x_1, c_2, \ldots, c_N) \in \Omega$ for $a_1 \le x_1 \le b_1$ by the previous step. Also, recall that Ω is a bounded, open, convex, connected set, so that $(x_1, x_2, c_3, \ldots, c_N) \in \Omega$ on the intervals of integration. And $G_2(c_2) = 0$.

By the proof from Case 1 for N=1, if there exist y_2, z_2 such that $y_2 < c_2 < z_2$ and $G_2(y_2)$, $G_2(z_2)$ have the same sign or $G_2(y_2) = G_2(z_2) = 0$, then there exist a_2, b_2 such that $a_2 < c_2 < b_2$ and $G_2(a_2) = G_2(b_2)$. It immediately follows that

$$\int_{a_2}^{b_2} \int_{a_1}^{b_1} g(x_1, x_2, c_3, \ldots c_N)\,dx_1 dx_2 = 0.$$

Next, for j=3,...,N we define

$$G_j(t) = \int_{c_j}^{t} \int_{a_{j-1}}^{b_{j-1}} \int_{a_{j-2}}^{b_{j-2}} \cdots \int_{a_1}^{b_1} g(x_1, x_2, \ldots, x_j, c_{j+1}, \ldots c_N)\,dx_1 dx_2 \ldots dx_j$$

for t such that $(x_1, x_2, \ldots, x_{j-1}, t, c_{j+1}, \ldots, c_N) \in \Omega$ for $a_i \le x_i \le b_i$, $i = 1,2,\ldots, j-1$. Note that $(x_1, x_2, \ldots, x_{j-1}, c_j, c_{j+1}, \ldots, c_N) \in \Omega$ for $a_i \le x_i \le b_i$, $i = 1,2,\ldots, j-1$ by the previous steps. Also, recall that Ω is a bounded, open, convex, connected set, so that $(x_1, x_2, \ldots, x_{j-1}, x_j, c_{j+1}, \ldots, c_N) \in \Omega$ on the intervals of integration. And $G_j(c_j) = 0$.

By the proof from Case 1 for N=1, if there exist y_j, z_j such that $y_j < c_j < z_j$ and $G_j(y_j)$, $G_j(z_j)$ have the same sign or

$G_j(y_j) = G_j(z_j) = 0$, then there exist a_j, b_j such that $a_j < c_j < b_j$ and $G_j(a_j) = G_j(b_j)$. It immediately follows that

$$\int_{a_j}^{b_j}\int_{a_{j-1}}^{b_{j-1}}\int_{a_{j-2}}^{b_{j-2}}\cdots\int_{a_1}^{b_1} g(x_1, x_2, \ldots, x_j, c_{j+1}, \ldots c_N)\,dx_1 dx_2 \ldots dx_j = 0.$$

When j = N, we obtain

$$\int_{a_N}^{b_N}\int_{a_{N-1}}^{b_{N-1}}\int_{a_{N-2}}^{b_{N-2}}\cdots\int_{a_1}^{b_1} g(x_1, x_2, \ldots, x_N)\,dx_1 dx_2 \ldots dx_N = 0.$$

Since $g(\vec{x}) = f(\vec{x}) - f(\vec{c})$, where $\vec{x} = (x_1, x_2, \ldots, x_N)$, re-arranging terms in the above identity yields

$$f(\vec{c}) = \frac{1}{\prod_{j=1}^{N}(b_j - a_j)}\int_{a_N}^{b_N}\int_{a_{N-1}}^{b_{N-1}}\int_{a_{N-2}}^{b_{N-2}}\cdots\int_{a_1}^{b_1} f(x_1, x_2, \ldots, x_N)\,dx_1 dx_2 \ldots dx_N .$$

Therefore, the set $S = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_N, b_N]$ satisfies

$$f(\vec{c}) = \frac{1}{|S|}\int_S f(\vec{x})\,d\vec{x} \quad \text{and } \vec{c} = (c_1, c_2, \ldots, c_N) \in S.$$

This completes the proof of Theorem 2.

† *Diane Denny, Ph.D.,* Texas A&M University-Corpus Christi, USA

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