The Median of a Generalized Pearson Distribution and its Relation with a Ramanujan-Equation

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Abstract

In this note we present the estimates for the median of various classes of Shakil-Kibria-Singh(SKS) distribution which is the solution of the Generalized Pearson Differential Equation(GPDE). Finally we show that the median of SKS distribution is connected to a Ramanujan sequence θ_n which is the solution to a famous Ramanujan equation.

$$\frac{e^n}{2} = \sum_{k=0}^{n} \frac{n^k}{k!} + \frac{n^n}{n!}(\theta_n - 1), \quad n = 0,1,2,\cdots$$

1. Introduction

A continuous distribution belongs to the Pearson system if its pdf (probability density function) f satisfies a differential equation of the form

$$\frac{1}{f(x)} \frac{d}{dx} f(x) = -\frac{x+a}{bx^2 + cx + d}, \tag{1.1}$$

where $a, b, c, d \in R$ are real parameter such that $f(x)$ is a probability density function of some continuous random variable X. Recently some researchers have considered the generalization of Pearson differential equation (1) as

$$\frac{1}{f(x)} \frac{d}{dx} f(x) = \frac{\sum_{j=0}^{m} a_j x^j}{\sum_{j=0}^{n} b_j x^j} \tag{1.2}$$

where $a_j, b_j \in R$ are real numbers and $m, n \in Z_+$ are positive integers. The differential equation (1.2) is called the Generalized Pearson Differential Equation (GPDE). The set

$$S = \left\{ f(x) \geq 0 : \frac{1}{f(x)} \frac{d}{dx} f(x) = \frac{\sum_{j=0}^{m} a_j x^j}{\sum_{j=0}^{n} b_j x^j}, a_j, b_j \in R; m, n \in Z_+ \right\} \text{ of}$$

probability density functions $f(x)$ is called generalized Pearson System.

Shakil-Kibria-Singh [5] consider GPDE (1.2) with

$$m = 2p, n = p+1, a_j = 0, j = 1, \cdots p-1, p+1, \cdots 2p-1; b = 0, j = 0, 1, \cdots, p, b_{p+1} \neq 0 \, and \, x > 0.$$

The solution to this special case of GPDE (1.2) is given as

$$f(x) = Cx^{\nu-1} \exp(-\alpha x^p - \beta x^{-p}), \quad x > 0, \alpha, \beta \geq 0, \nu \in R \tag{1.3}$$

$$\alpha = -\frac{a_{2p}}{pb_{p+1}}, \beta = \frac{a_0}{pb_{p+1}}, \nu = \frac{a_p + b_{p+1}}{b_{p+1}}, b_{p+1} \neq 0, \; p \in Z_+, C \text{ is a}$$

normalizing constant is called Shakil-Kibria-Singh (SKS) distribution, see Shakil et al. [5] and Hamedani [4]. The characterization of SKS distribution has been obtained in Hamedani [3]. As pointed out by Shakil et al. [5], the SKS distribution can be classified into following three classes:

Class I: $\alpha > 0$, $\beta = 0$, $\nu > 0$, and $p > 0$;

$$C = \frac{p \, (\alpha)^{\frac{\nu}{p}}}{\Gamma\left(\frac{\nu}{p}\right)}.$$

(1.4)

Class II: $\alpha = 0$, $\beta > 0$, $\nu < 0$, and $p > 0$;

$$C = \frac{p}{(\beta)^{\frac{\nu}{p}} \Gamma\left(-\frac{\nu}{p}\right)}.$$

(1.5)

Class III: $\alpha = 0$, $\beta > 0$, $\nu < 0$, and $p > 0$;

$$C = \frac{p}{(\beta)^{\frac{\nu}{p}} \Gamma\left(-\frac{\nu}{p}\right)}.$$

(1.6)

In this note we develop the estimates for the median for the various classes of SKS distribution and show that the median of the gamma distribution is a special case of SKS distribution. Further, we have shown that the median of SKS distribution is remarkably connected with the Ramanujan sequence θ_n, namely,

$$\theta_n = 1 - \left(\frac{e}{n}\right)^n \int_n^{\alpha\lambda} e^{-t} t^n dt$$

(1.7)

where λ is the median of SKS distribution and θ_n is the solution of the famous Ramanujan equation

$$\frac{e^n}{2} = \sum_{k=0}^{n} \frac{n^k}{k!} + \frac{n^n}{n!}(\theta_n - 1), \quad n = 0,1,2,\cdots \quad , \frac{1}{3} \le \theta_n \le \frac{1}{2} \ .$$

(1.8)

2. The Median of SKS Distribution

The median of a random variable X is defined as a number $m \in R$ such that

$$P(X \le m) \ge \frac{1}{2} \; and \; P(X \ge m) \ge \frac{1}{2}.$$

For an absolute continuous probability distribution with the pdf f, the median m satisfies the equation

$$P(X \le m) = P(X \ge m) = \int_{-\infty}^{m} f(x)dx = \frac{1}{2}$$

For Class I, we have $\alpha > 0, \beta = 0, \nu > 0 \quad and \; p \in Z_+$ and

$$f(x) = Cx^{\nu-1}e^{-\alpha x^p}, \; where \; C = \frac{p\alpha^{\frac{\nu}{p}}}{\Gamma\left(\frac{\nu}{p}\right)},$$ and Z_+ is the set of all positive integers.

Let λ be the median of SKS distribution. Then, from the above definition of the median, λ is uniquely determined by the equation

$$\int_{0}^{\lambda} f(x)dx = \frac{1}{2}$$

That is,

$$\frac{p\alpha^{\frac{\nu}{p}}}{\Gamma\left(\frac{\nu}{p}\right)} \int_{0}^{\lambda} x^{\nu-1}e^{-\alpha x^p}dx = \frac{1}{2}$$

Now, using Eq. (2.1), we will establish various results which are provided below.

Case (2.1) We have

$$e^{-\alpha x^p} = 1 + \frac{(-\alpha x^p)}{1!} + \frac{(-\alpha x^p)^2}{2!} + \frac{(-\alpha x^p)^3}{3!} + \cdots$$

Then $\quad x^{\nu-1}e^{-\alpha x^p} = x^{\nu-1} - \frac{\alpha x^{p+\nu-1}}{1!} + \frac{\alpha^2 x^{2p+\nu-1}}{2!} - \frac{\alpha^3 x^{3p+\nu-1}}{3!} + \cdots$

Hence, from Eq. (2.1), we have

$$\frac{p\alpha^{\frac{v}{p}}}{\Gamma\left(\frac{v}{p}\right)}\int_0^{\lambda}\left\{x^{(v-1)}-\left(\frac{\alpha x^{(p+v-1)}}{1!}\right)+\left(\frac{\alpha^2 x^{(2p+v-1)}}{2!}\right)-\left(\frac{\alpha^3 x^{(3p+v-1)}}{3!}\right)+\cdots\right\}dx =$$

$$\frac{p\alpha^{\frac{v}{p}}}{\Gamma\left(\frac{v}{p}\right)}\left\{\left(\frac{x^{(v)}}{v}\right)-\frac{\alpha x^{(p+v)}}{(p+v)1!}+\frac{\alpha^2 x^{(2p+v)}}{(2p+v)2!}-\frac{\alpha^3 x^{(3p+v)}}{(3p+v)3!}+\cdots\right\}_0^{\lambda} = \frac{p\alpha^{\frac{v}{p}}}{\Gamma\left(\frac{v}{p}\right)}\left\{\frac{\lambda^v}{v}-\frac{\alpha\lambda^{(p+v)}}{(p+v)1!}+\frac{\alpha^2\lambda^{(2p+v)}}{(2p+v)2!}-\frac{\alpha^3\lambda^{(3p+v)}}{(3p+v)3!}+\cdots\right\}$$

$=\frac{1}{2}$, which implies

$$\sum_0^{\infty}\frac{\alpha^{2n}\lambda^{(2np+v)}}{(2np+v)n!}-\sum_0^{\infty}\frac{\alpha^{2n+1}\lambda^{(2n+1)p+v}}{[(2n+1)p+v](2n+1)!}=\frac{1}{2} \tag{2.2}$$

Case (2.2): Let $v = p$

In equation (2.1), we have

$$\frac{p\alpha}{\Gamma(1)}\int_0^{\lambda} x^{p-1}e^{-\alpha x^p}\,dx=\frac{1}{2}$$

That is,

$$\frac{p\alpha}{\Gamma(1)}\int_0^{\lambda}\frac{-\alpha p x^{p-1}e^{-\alpha x^p}}{-\alpha p}\,dx=\frac{1}{2}$$

Since $-\alpha p x^{p-1}e^{-\alpha x^p}$ is the derivative of $-e^{-\alpha x^p}$, so on simplifying and evaluating above integral, we get $e^{-\alpha\lambda^p}=\frac{1}{2}$.

That is,

$$\lambda=\left[\frac{\ln 2}{\alpha}\right]^{\frac{1}{p}} \tag{2.3}$$

Case (2.3) Let v=1. Then from (2.1) we have

$$\frac{p\alpha^{\frac{1}{p}}}{\Gamma\left(\frac{1}{p}\right)}\int_0^{\lambda} e^{-\alpha x^p}\,dx = \left(\frac{1}{2}\right).$$

In particular, when $p = 2$, we have

$$\frac{p\alpha^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)}\int_0^{\lambda} e^{-\alpha x^2}\,dx = ½.$$

That is,

$$\int_0^{\lambda} e^{-\alpha x^2}\,dx = \frac{\Gamma\left(\frac{1}{2}\right)}{4\alpha^{\frac{1}{2}}}.$$

Substituting y $=\sqrt{\alpha}\,x$, then above integral reduces to

$$\sqrt{\alpha}\int_0^{\sqrt{\alpha}\lambda} e^{-y^2}\,dy = \frac{\Gamma\left(\frac{1}{2}\right)}{4\alpha^{\frac{1}{2}}}$$

That is,

$$\int_0^{\sqrt{\alpha}\lambda} e^{-y^2}\,dy = \frac{\sqrt{\pi}}{4\alpha}.$$

(2.4)

Case (2.4): From (2.1), when p =1, we have

$$\frac{\alpha^{\upsilon}}{\Gamma(\upsilon)}\int_0^{\lambda} x^{\upsilon-1}e^{-\alpha x}\,dx = \frac{1}{2}\ .$$

Let $\alpha x = t$. Then $dx = \frac{1}{\alpha}dt$ and $x^{\nu-1} = \frac{t^{\nu-1}}{\alpha^{\nu-1}}$, so that

$$\frac{\alpha^{\upsilon}}{\Gamma(\upsilon)}\int_0^{\alpha\lambda} \frac{t^{\upsilon-1}}{\alpha^{\upsilon-1}}e^{-t}\frac{1}{\alpha}\,dt = \frac{1}{2}.$$

That is,

$$\int_0^{\alpha\lambda} x^{\upsilon-1}e^{-x}\,dx = \frac{\Gamma(\upsilon)}{2}.$$

(2.5)

For $\nu = 1$, the Eq. (2.5) reduces to

$$\frac{\alpha}{\Gamma(1)}\int_0^{\alpha\lambda} e^{-t}\frac{1}{\alpha}\,dt = \int_0^{\alpha\lambda} e^{-t}\,dt = \frac{e^{-t}}{-1}\Big]_0^{\alpha\lambda} = \{1 - e^{-\alpha\lambda}\} = \frac{1}{2},$$

which gives

$$\lambda = \frac{\ln(2)}{\alpha} \tag{2.6}$$

Case (2.5) From Eq. (2.1), for p = 2 and $\nu = 1$, we have

$$\frac{2\alpha^{\frac{1}{2}}}{\Gamma(\frac{1}{2})}\int_0^{\lambda} e^{-\alpha x^2}\,dx = 2\sqrt{\frac{\alpha}{\pi}}\int_0^{\lambda} e^{-\alpha x^2}\,dx = \frac{1}{2}.$$

From the above equation, using the definition of error function, that is,

$\mathrm{erf}(z) = \dfrac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dz$, we obtain the following equation:

$$\mathrm{erf}(\lambda\sqrt{\alpha}) = \frac{1}{2\sigma\sqrt{2\alpha}},$$

which can be solved numerically for λ, for different values of the parameters α and σ. For example, (i) when $\alpha = 1$ and $\sigma = 1$, we have $\lambda \approx 0.32$, and (ii) when $\alpha = 2$ and $\sigma = 1$, we have $\lambda \approx 0.103$.

Again, if $n = \nu - 1$, and, in particular, ν is a positive integer, then, from Eq. (2.5), we have

$$\frac{1}{n!}\int_0^{\alpha\lambda} x^n e^{-x}\,dx = 1/2 \tag{2.7}$$

We know that the median λ_n of the gamma distribution of order $n+1$ with parameter 1 is given as

$$\frac{1}{n!}\int_0^{\lambda_n} x^n e^{-x}\,dx = \frac{1}{2} \tag{2.8}$$

It follows from (2.7) and (2.8)

$$\alpha\lambda = \lambda_n .$$

(2.9)

As discussed in Alzer [2] and in view of equation (2.9) the value of the median of SKS distribution satisfies the inequality

$n+\frac{2}{3} < \alpha\lambda \le \min\left(n + \log 2,\ \frac{2}{3} + \frac{1}{2(n+1)}\right)$, where n = 0, 1 ,2, ...

(2.10)

and the asymptotic expansion for $\alpha\lambda$ is given as

$$\alpha\lambda = n + \frac{2}{3} + \frac{8}{405n} - \frac{64}{5103n^2} + \frac{2944}{492075n^3} + O\left(\frac{1}{n^4}\right)$$

(2.11)

Using Eq. (2.11), we have computed some values of $\alpha\lambda$, which are provided in the following table.

$n=\nu-1$	$\alpha\lambda$	$n=\nu-1$	$\alpha\lambda$	$n=\nu-1$	$\alpha\lambda$
1	1.679860939…	6	6.669638167…	11	11.66836325…
2	2.674155653…	7	7.669250027…	12	12.66822912…
3	3.672079099…	8	8.668951525…	13	13.66811465…
4	4.670914567…	9	9.668714826…	14	14.66801579…
5	5.670163481…	10	10.66852254…	…….	……

Case (2.6) In Class II when $\alpha = 0, \beta > 0, \nu < 0$ and $p \in Z_+$,

$$f(x) = \frac{p}{\beta^{\frac{\nu}{p}}\,\Gamma\left(-\frac{\nu}{p}\right)} x^{\nu-1} e^{-\beta x^{-p}}$$

Now $\int_0^{\lambda} f(x)\,dx = \frac{1}{2}$, implies

$$c_2 \int_0^{\lambda} x^{\nu-1} e^{-\beta x^{-p}}\,dx = \frac{1}{2},\ \ where,\ c_2 = \left(\frac{p}{\beta^{\frac{\nu}{p}}\Gamma\left(-\frac{\nu}{p}\right)}\right)$$

When we consider p =1,

$$\left(\frac{C_2}{\beta}\right)\left\{\lambda^{\nu+1}e^{-\frac{\beta}{\lambda}}-\left(\frac{\nu+1}{\beta}\right)\left\{\lambda^{(\nu+2)}e^{\left(-\frac{\beta}{\lambda}\right)}-\left(\frac{\nu+2}{\beta}\right)\left\{\lambda^{(\nu+3)}e^{\left(-\frac{\beta}{\lambda}\right)}\ldots-\left\{\nu+\right.\right.\right.\right.$$

$$\left.\left.\left.\left.(-\nu+2)\left\{\lambda^{(\nu+(-\nu+2))}e^{\left(-\frac{\beta}{\lambda}\right)}-\{\nu+(-\nu+1)\}\left(\frac{e^{\left(-\frac{\beta}{\lambda}\right)}}{\beta}\right)\right\}\right\}\right\}\right\}\right\}$$

= ½

(2.12)

For the case ν=-1, p =1, we have

$C_2 = \left(\frac{p}{\beta^{\left(\frac{\nu}{p}\right)}\Gamma\left(-\frac{\nu}{p}\right)}\right) = \beta$, $\lambda = \left(\frac{\beta}{\ln\left(\frac{2C_2}{\beta}\right)}\right)$, $\lambda = \frac{\beta}{\ln(2)}$

(2.13)

Case (2.7) Class III when α > 0, β >0, υ ε R and $p \in Z_+$

f(x) = $C_3 x^{\nu-1} e^{-\alpha x^p - \beta x^{-p}}$**,** $C_3 = \left(\frac{p}{2}\right)\left(\frac{\alpha}{\beta}\right)^{\frac{\nu}{2p}} \frac{1}{K_{\left(\frac{\nu}{p}\right)}(2\sqrt{\alpha\beta})}$

Where $K\left(\frac{\nu}{p}\right)\left(2\sqrt{\alpha\beta}\right)$ represent the modified Bessel Function of third Kind

Median: $\int_0^{\lambda} f(x)\,dx = \frac{1}{2}$, $C_3\int_0^{\lambda} x^{\nu-1} e^{-\alpha x^p - \beta x^{-p}}$ dx=$\frac{1}{2}$

When p=1, α=β , υ=-1, then:

f(x)= $\left(\frac{1}{2K_{\nu(2\alpha)}}\right)x^{-2} e^{-\alpha(x^p - x^{-p})}$,

then

$\frac{1}{2\alpha K_{\nu(2\alpha)}}\left\{1 - e^{-\alpha\lambda - \frac{\alpha}{\lambda}} - \alpha\int_0^{\lambda} e^{-\alpha x - \frac{\alpha}{x}}\,dx\right\} = \frac{1}{2}$,

For p≫1 ,ν=p ,

λ^p= - $\frac{1}{\alpha}$ ln(1- $\frac{\alpha p}{2C_3}$)

(2.14)

3. Ramanujan Problem and its relation with the median of SKS distribution

The following problem was posed by Ramanujan [6]: for $n = 1,2,3,\ldots$ show that

$$\frac{e^n}{2} = \sum_{k=0}^{n} \frac{n^k}{k!} + \frac{n^k}{k!}(\theta_n - 1), \qquad \theta \in [1/3, 1/2] \tag{3.1}$$

The above Ramanujan problem (3.1) has attracted the attention of many researcher see for example various references cited in [1, 2].

We have

$$\frac{e^n}{2} = \sum_{k=0}^{n} \frac{n^k}{k!} + \frac{n^n}{n!}(\theta_n - 1)$$

i.e.,
$$\frac{n!}{n^n}\left(\frac{e^n}{2} - \sum_{k=0}^{n} \frac{n^k}{k!}\right) = (\theta_n - 1)$$

i.e.,
$$\theta_n = \frac{n!}{n^n}\left(\frac{e^n}{2} - \sum_{k=0}^{n-1} \frac{n^k}{k!}\right) \tag{3.2}$$

As observed in Alzer [2] the median λ_n of gamma distribution is connected with Ramanujan sequence θ_n as follows:

$$1 - \theta_n = \left(\frac{e}{n}\right)^n \int_n^{\lambda_n} e^{-t} t^n dt \tag{3.3}$$

Now it follows from equations (2.9) and (3.3) the median of SKS distribution λ is connected with the Ramanujan sequence θ_n as

$$\theta_n = 1 - \left(\frac{e}{n}\right)^n \int_n^{\alpha\lambda} e^{-t} t^n dt$$

(3.4)

, where $\alpha > 0$ is a parameter in SKS distribution.

4. Concluding Remarks

For class I of the SKS distribution, the probability density function f is given as

$$f(x) = \frac{p\alpha^{\nu/p}}{\Gamma(\nu/p)} \int_0^{\lambda} x^{\nu-1} e^{-\alpha^p} dx.$$

With some particular choice of parameters, such as considering ν as a positive integer and taking $n = \nu - 1$, we prove that the median of the gamma distribution is a special case of the Median of SKS distribution when parameter α is chosen as 1. In view of this relation various properties developed for the median of the gamma distribution will be automatically applicable for the median of SKS distribution. It is the subject of further investigation that the median estimated in various other cases of SKS distribution can reveal some close connection between the probability theory and the number theory like class I of SKS distribution.

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