Prospective Elementary Teachers' Explicit Knowledge of the Arbitrary Nature of the Unit within Contexts Involving Fractions

José N. Contreras, Ph. D. †
Armando M. Martínez-Cruz Ph. D. ‡

Abstract

This study reports an analysis of 29 prospective elementary teachers' explicit knowledge of the arbitrary nature of the unit within contexts involving fractions. To this end, the participants were administered two questionnaires. The first questionnaire included an open-response task in which they were asked what fraction or fractions the shaded portion of a diagram could represent. The second questionnaire consisted of six tasks that asked the subjects to justify whether the same picture could be used to represent the following fractions ¾, 3/5, 3/10, 1 ½, ½ and 1. Results indicate that most students were not able to conceptualize that a given shaded region could represent different fractions from the ones most naturally suggested by the given diagram.

Introduction

Research on teachers' knowledge has evolved from examining teacher's general knowledge of mathematics to teachers' knowledge of specific mathematical content (Ball, 1990; Ball, Lubienski, & Mewborn, 2001; Davis & Simmt, 2006, Graeber & Tirosh, 1988). This evolution was the result of the failure of past research in finding a link between teachers' knowledge and student achievement. Shulman (1986) was one of the first researchers to argue that teachers need to have pedagogical content knowledge, a special type of knowledge that includes students' conceptions and misconceptions, typical difficulties that students have, and multiple representations of a particular topic.

Building on Shulman's construct of pedagogical content knowledge, Ball and her colleagues (Ball & Bass, 2000; Ball, Lubienski, & Mewborn, 2001; Hill Schilling, and Ball, 2004) have used the term "mathematical knowledge for teaching" to refer to the mathematics knowledge that teachers use when performing instructional tasks to teach a specific mathematical topic, such as the concept of fraction.

Past research has documented that teachers' knowledge of fractions and operations is underdeveloped (Graeber, Tirosh, & Glover, 1989; Izsák, 2008; Simon, 1993; Tirosh & Graeber, 1989; Tobias, 2013). Izsák (2008), for example, examined the mathematical knowledge that two sixth-grade teachers used when teaching lessons on multiplication of fractions. He concluded that for teachers to teach effectively multiplication of fractions, they need to coordinate flexibly three-level unit structures. In a more recent study, Tobias (2013) investigated the difficulties that some prospective elementary teachers had with identifying the referent whole and corresponding language that they used for describing fractional amounts. It is then imperative that researchers investigate prospective teachers' explicit knowledge of the unit within contexts involving fractions. This study focuses mainly on characterizing prospective elementary teachers' explicit knowledge of the arbitrary nature of the unit within fractional contexts involving drawn diagrams. However, through examination of this characterization, this study may also contribute to our continuing discussion of what might constitute mathematical knowledge for teaching the concept of fraction.

Empirical Background

Concepts related to fraction and their associated operations, particularly multiplication and division, are challenging for most students to learn (Lamon, 2005; Mack, 1995; Ni & Zhou, 2005; Streefland, 1991). One of the most fundamental ideas in developing fractional thinking is the concept of the unit or whole. However, researchers have documented that students and prospective elementary teachers have difficulties conceptualizing the whole within problem situations involving fractions (Ball, 1990; Luo, Lo, & Leu, 2011; Simon, 1993; Tobias, 2013). Ball, for example, asked 19 prospective elementary and secondary teachers to write a word problem for which 1 ¾ ÷ ½ is the appropriate mathematical computation. She reported that only five participants were able to generate an appropriate word problem. The remaining 14 college students either generated an inappropriate representation or were unable to generate a representation. Several of the inappropriate story problems could be solved using 1 ¾ ÷ 2 whose solution is 7/8. When comparing the two solutions (3 ½ for 1 ¾ ÷ ½ and 7/8 for 1 ¾ ÷ 2) some prospective teachers used a context involving pizzas and reasoned that each pizza "is divided into four pieces, so you have seven pieces. So each person gets 7/8 of a pizza, which is 3 ½ pieces of pizza." Notice that 3 ½ pieces of pizza is refereeing to 3 ½ fourths of a pizza as opposed to 3 ½ one-halves of a pizza. As another example, Simon (1993) presented 33 prospective elementary teachers with the task of finding how much flour is left over after one makes as many cookies as possible with 35 cups of flour if each cookie requires 3/8 of a cup. He reported that only about 15% of the students provided a correct response while about 30% of the respondents claimed that 1/3 of a cup of flour was left over. The remaining prospective elementary teachers provided other incorrect solutions or no solutions.

Conceptual Framework

Some challenges that students have in learning concepts associated with fractions result from incorrectly applying ideas from whole numbers. In the context of whole numbers, the unit is always explicit (e.g., 7 always refers to 7 units). Within contexts involving fractions, on the other hand, the unit is often implicit. By its own nature, a fraction is always linked to a unit, so one must always identify the unit in a contextual problem situation (what is "one whole" object or set?). Unlike with whole numbers, a unit within fractional contexts may not necessarily be a discrete set of objects. The nature of the unit may change not only across contexts but also within the same context. For example, one can of cola may be thought of as 1/24 of a case, 1/12 of a dozen, 1/6 of a 6-pack, etc. It is the flexibility to reconceptualize quantities in different chunks that helps learners to link meanings, symbols, and operations so they can apply their knowledge in different contexts (Lamon, 2002).

The mental process of assigning a unit of measurement to a given quantity is known as unitizing (Lamon, 1996, 2002). It can be thought of as the size chunk one must construct "in terms of which to think about a given commodity" (Lamon, 1996, p. 170). Of course, the unitizing process is also reversible. Given a fraction represented with a diagram, one may need to identify the unit or whole.

Previous research has provided evidence that changes on the nature of the unit within a problem situation explains some of the conceptual hurdles that students experience in linking procedural and conceptual fractional knowledge (Behr, Harel, Post, & Lesh, 1992; Harel & Confrey, 1994; Hiebert & Behr, 1988; Lamon, 1996). More recently, researchers have concluded that the challenges in coordinating flexibly different levels of unit structures accounts to a great extent for the cognitive difficulties in understanding multiplication and division of fractions (Izsák, 2008; Izsák, Jacobson, de Araujo, & Orill, 2012).

Methodology and Data Sources

Twenty nine prospective elementary teachers enrolled in a mathematics content course for elementary education majors participated in this study. They were asked to complete two questionnaires with a total of 7 tasks. The first questionnaire consisted of only one task (Fig. 1) and the second questionnaire consisted of 6 tasks (Figure 2). After all students completed questionnaire 1 they were given questionnaire 2.

The tasks included in the questionnaire were designed to assess prospective elementary teachers' explicit knowledge of the arbitrary nature of a unit within fraction contexts. By formulating the problem "in reverse" – by providing a diagram with a shaded portion and asking them the fraction or fractions that the shaded part of the figure could represent or whether the shaded diagram could represent certain fractions – there was a demand for an explicit awareness of the arbitrary nature of a unit.

Results

A content analysis for each student' response was performed. Table 1 summarizes the results for the task included in questionnaire 1. As we can see from the table, a majority of the students (about 62%) thought that the shaded portion of the diagram could represent (only) ¾.

Five (about 17%) students thought that that the shaded portion of the diagram could represent 3/5. An analysis of their writing responses revealed that all of them thought of the diagram as a complete circle: there are 3 out of 5 (equal) shaded parts. Five students thought that the shaded portion of the diagram could represent either ¾ or 3/5 depending on whether there were 4 or 5 (equal) parts. One student thought of the diagram representing 3/5 or 4/5 depending on whether the "missing" part was shaded or not. It is interesting to notice that all students justified their responses based on the idea that the unit or whole needs to be physically present.

To gain further insights into prospective elementary teachers' conceptions of the nature of the unit, they were asked to complete questionnaire 2. The results of the analysis are displayed in table 2.

As expected, most of the students (27 or 93%) stated that the shaded portion of the diagram could be used to represent ¾. One of the students who said "no" wrote that "3/4 doesn't make a whole according to this figure. The answer should be 3/5" while the other drew a circle and shaded ¾ of it arguing that this last representation "works better."

Regarding whether the shaded portion of the diagram could be used to represent 3/5, 17 (59%) responded affirmatively while 11 (38%) students said "no" and one wrote "yes and no." All of the students who said "yes" wrote that the circle needs to be completed. None of them conceptualized the shaded portion of the diagram as representing 3/5 without the fifth portion being physically present. Most of the students who said "no" wrote that there were only four parts, 3 of which were shaded. Finally, the student who wrote "yes and no" claimed that it depended on whether the "missing" piece was shaded or not: If the piece was not shaded then the shaded portion of the diagram would represent 3/5, otherwise the shaded portion would represent 4/5.

As to the third possibility, all students wrote that the shaded portion of the diagram could not be used to represent 3/10. The typical explanation was that there were only four sections, not 10. Some students wrote that they could get 10 pieces (by completing the circle and dividing every section into 2 parts), but that in this case the shaded portion of the diagram would represent 6/10, not 3/10.

The fourth task in questionnaire 2 asked students whether the shaded portion of the diagram could be used to represent 1 ½. As indicated in table 2, most of the students wrote "no" while 7 (about 24%) students responded "yes." One student did not provide a response. To gain a deeper understanding of the cognitive processes that students used to respond to the task, we examined their written justifications. Here is a typical example of students' reasoning to justify a negative response: "No, there is only 1 figure, so it cannot be > than 1." It is interesting to mention that two of the seven affirmative responses contained incorrect explanations. Both justifications used a part to part comparison: 3 shaded parts to 2 non-shaded parts. A typical correct explanation is the following: "Only if each part represents 1/2. In that case there are 3 parts that are shaded and 1/2 + 1/2 + 1/2 = 1 ½."

In contrast to the case for 1 ½, only 3 (10%) students wrote that the shaded portion of the diagram could be used to represent ½. However, the three explanations were incorrect because the three students changed the shaded portion. For example, one student said that "if two (parts) of this diagram represent 1 then 1 (part) of this diagram would represent ½. A typical justification for a negative response was that the shaded portion of the diagram was more than ½ shaded.

The last task asked students a similar question for 1. As displayed in table 2, 6 (21%) students said that the shaded portion of the given diagram could be used to represent 1 whereas 22 (76%) wrote "no". One student did not provide a response. It is interesting to mention that three students wrote "yes if you shade all the parts" while one student said "yes, but the empty triangle would have to go away." Only two students provided a correct explanation. One of them wrote "yes, only if each part represent (*sic*) 1/3. In that case, there are 3 parts that are shaded and 1/3 + 1/3 + 1/3 = 1."

Discussion

To respond to the question of what fraction or fractions a shaded portion of the given diagram (Fig. 1) could represent, most of the students (62%) wrote ¾. This finding is not surprising because the diagram consists of 4 equal parts of which 3 are shaded. It is natural to think that the shaded portion represents ¾ of the whole: 3 out of 4 (equal) parts are shaded. Also, not surprisingly either, some students thought that the diagram could represent 3/5 assuming automatically that the given diagram was a circle, which they completed. It is worthwhile mentioning that all the students conceptualized the unit associated to the shaded portion of the diagram as being physically present: they could not conceptualize that the shaded portion of the diagram could represent ¾ or 3/5 by itself, without the need to have four or five equal parts physically present. For example, the diagram could represent ¾ (3/5) square units or ¾ (3/5) pounds of cake.

To further probe prospective elementary teachers' conscious knowledge of the arbitrary nature of the unit associated to a given picture, they were asked whether the shaded portion could represent ¾, 3/5, 3/10, 1 ½, ½, and 1. Almost all students (27 or 93%) answered affirmatively for ¾ while the majority (17 or 59%) wrote "yes" for 3/5. Again, those findings are not surprising because the given diagram consists of 4 equal parts out of which 3 are shaded. All the students who wrote that the shaded portion of the diagram could represent 3/5 justified their response by saying that the circle needs to be completed. To reiterate, the students were not able to conceptualize that the shaded portion of the diagram could represent 3/5 by itself, without the "missing" part being present. This interpretation is further supported by the fact that none of the students was able to think of a situation for which the shaded diagram could represent 3/10. For the picture to represent 3/10, they needed to physically see 3 parts out of 10. The students had the three shaded equal parts, another non-shaded part, another missing part, but the remaining 5 parts that they needed to think of the shaded portion as 3/10 were not physically present. The findings for the case ½ further substantiate our interpretation: None of the students were able to reconceptualize the three shaded equal parts of the diagram as ½ because they did not have physically present the other 2 parts.

As shown by the results regarding the cases 1 ½ and 1, the explanation that the unit needs to be physically present is not enough to understand our findings. In both cases all the parts needed to reconceptualize the shaded portion of the diagram as 1 ½ or 1 were physically present. Five students were able reconceptualize that the shaded portion of the diagram could be used to represent 1 ½ because they saw that each piece could represent ½. Even though all the parts to reinterpret the shaded portion of the diagram were physically present, the other students were not able to "see" the chunks needed to think of the shaded portion as 1 ½. A similar explanation accounts for the fact that only two students provided a correct explanation to reconceptualize the shaded portion as 1. We contend that these findings can be explained by the idea that not all students have completely developed the ability to spontaneously reunitize, that is, the capacity to automatically reconceptualize a given quantity in different chunks of different sizes.

In summary, the findings of this study seem to indicate that prospective elementary teachers are not able to reconceptualize a given diagram in terms of different fractions due to two factors: the referent unit is not physically present and their ability to reunitize is underdeveloped. They seem to need instructional experiences to expand their conceptions of fractions representing quantities, to better understand the arbitrary nature of the unit, and to reconceptualize a given quantity in terms of different-sized chunks. We can conclude that further research is needed to gain a more profound understanding of teachers' knowledge about the arbitrary nature of the unit within contexts involving fractions.

† *José N. Contreras, Ph. D.,* Ball State University, USA
‡ *Armando M. Martínez-Cruz, Ph. D.,* California State University-Fullerton, USA

What fraction or fractions could the shaded part of the figure represent? Justify your responses and draw diagrams if appropriate.

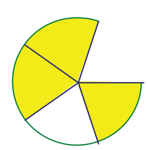
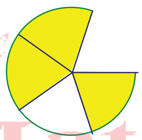


Figure 1: Task included in questionnaire 1

1. Could the shaded part of the following diagram be used to represent $\frac{3}{4}$? Justify your response. If appropriate, draw the diagram that represents 1.



[Similar questions were asked for $\frac{3}{5}$, $\frac{3}{10}$, $1\frac{1}{2}$, $\frac{1}{2}$, and 1]

Figure 2: Tasks included in questionnaire 2

Response	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{3}{4}$ or $\frac{3}{5}$	$\frac{3}{5}$ or $\frac{4}{5}$
Frequency (Percentage)	18 (62%)	5 (17%)	5 (17%)	1 (3%)

Table 1: Results for questionnaire 1

	¾	3/5	3/10	1 ½	½	1
Yes	27 (93%)	17 (59%)	0 (0%)	7 (24%)	3 (10%)	6 (21%)
No	2 (7%)	11 (38%)	29 (100%)	21 (72%)	25 (86%)	22 (76%)
Yes and No		1 (3%)				
No response				1 (3%)	1 (3%)	1 (3%)

Table 2: Results for tasks included in questionnaire 2

References

Ball, D. L. (1990). Prospective elementary and secondary teachers' understanding of division. *Journal for Research in Mathematics Education*, 21(2), 132–144.

Ball, D. L., & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 81–104). Westport, CT: Ablex.

Ball, D., Lubienski, S., & Mewborn, D. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), *Handbook of research on teaching* (4th ed., pp. 433–456). Washington, DC: American Educational Research Association.

Behr, M. J., Harel, G., Post, T., & Lesh, R. (1992). Rational number, ratio, and proportion. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 296–333). New York: Macmillan.

Davis, B., & Simmt, E. (2006). Mathematics for teaching: An ongoing investigation of the mathematics that teachers (need) to know. *Educational Studies in Mathematics*, 30(3), 213–228.

Graeber, A., Tirosh, D., (1988). Multiplication and division involving decimals: Preservice elementary teachers' performance and beliefs. *Journal of Mathematical Behavior*, 7, 263–280.

Graeber, A., Tirosh, D., & Glover, R. (1989). Preservice teachers' misconceptions in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Ediucation*, 20(1), 95–102.

Harel, G., & Confrey, J. (Eds.). (1994). *The development of multiplicative reasoning in the learning of mathematics*. Albany, NY: SUNY Press.

Hiebert, J., & Behr, M. (Eds.). (1988). *Number concepts and operations in the middle grades*. Reston, VA: National Council of Teachers of Mathematics.

Hill, H. C., Schilling, S. G., & Ball, D. L. (2004). Developing measures of teachers' mathematical knowledge for teaching. *Elementary School Journal*, 105, 11– 30.

Izsák, A. (2008). Mathematical knowledge for teaching fraction multiplication. *Cognition and Instruction*, 26(1), 95–143.

Izsák, A., Jacobson, E., de Araujo, Z., & Orrill, C. H. (2012). Measuring mathematical knowledge for teaching fractions with drawn quantities. *Journal for Research in Mathematics Education*, 43(4), 391–427.

Lamon, S. (1996). The development of unitizing: Its role in children's partitioning strategies. *Journal for Research in Mathematics Education*, 27(2), 170–193.

Lamon, S. (2002). Part-Whole comparisons with unitizing. In B. Litweller, & G Bright (Eds.), *Making sense of fractions, ratios, and proportions* (pp. 79–86). Reston, Va: National Council of Teachers of Mathematics Inc.

Lamon, S. J. (2005). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers*. Mahwah, NJ: Lawrence Erlbaum Associates.

Luo, F., Lo, J., & Leu, Y. (2011). Fundamental fraction knowledge of prospective elementary teachers: A cross-national study in the United States and Taiwan. *School Science and Mathematics*, 111(4), 164–177.

Mack, N. K. (1995). Confounding whole-number and fraction concepts when building on informal knowledge. *Journal for Research in Mathematics Education*, 26(5), 422–441.

Ni, Y., & Zhou, Y. D. (2005). Teaching and learning fraction and rational numbers: The origins and implications of whole number bias. *Educational Psychology*, 40(1), 27–52.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4–14.

Simon, M. A. (1993). Prospective elementary teachers' knowledge of division. *Journal for Research in Mathematics Education*, 24(3), 233–254.

Streefland, L. (1991). *Fractions in realistic mathematics education: A paradigm of developmental research*. The Netherlands: Kluwer Academic Publishers.

Tirosh, D., & Graeber, A. (1989). Preservice elementary teachers' explicit beliefs about multiplication and division. *Educational Studies in Mathematics*, 20, 79–96.

Tobias, J. M. (2013). Prospective elementary teachers' development of fraction language for defining the whole. *Journal of Mathematics Teacher Education*, 16(2), 85–103.