Critical Numbers and Zeros

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Abstract

Basic definitions fundamental to the paper are established. Preliminary material concerning the Generalized Product Rule for Derivatives is presented. A main theorem is developed which verifies that a polynomial p(x) of degree at least 3 with $\deg(p(x))$ distinct real zeros has a unique critical number between each consecutive pair of zeros and no other critical numbers. Furthermore, it is established that the smallest critical number is closer to the smallest zero than the next to smallest zero. Similarly, it is shown that the largest critical number is closer to the largest zero than the next to largest zero.

Introduction

In 1991 Moran posed the problem that if p(x) is a cubic polynomial with real zeros $\{a_i\}_{i=1}^{3}$, where $a_1 < a_2 < a_3$, then p(x) has a critical number c such that $a_1 < c < a_2$ and c is nearer to a_1 than a_2 [4, p. 344, Problem 459]. In 1992 Williams presented a solution to this specific problem [7, p. 345, Problem 459]. Waterhouse also produced a solution to this problem for a more general class of functions which included polynomials of degree $n \geq 3$ with distinct real zeros $\{a_i\}_{i=1}^{n}$, where $a_i < a_{i+1}$ for $1 \leq i \leq n-1$ [6, pp. 345-346, Problem 459]. The result by Waterhouse generalized that of Williams by showing that polynomials in this larger class also have a critical number c with the property that $a_1 < c < a_2$ and c is nearer to a_1 than a_2. This paper will establish that polynomials such as those included in the result by Waterhouse have a unique critical number $c_i \in (a_i, a_{i+1})$ for $1 \leq i \leq n-1$ and no other critical numbers. Furthermore, the result by Waterhouse is verified by using a different approach than Waterhouse to show that the smallest critical number c_1 is nearer to a_1 than a_2. Finally, a symmetrical fact involving the largest critical number c_{n-1} is then established by showing that c_{n-1} is nearer to a_n than a_{n-1}.

Preliminary Results

Suppose that f(x) is a real valued function of a real variable and c is a real number. Using the standard definition, c is a critical number (or critical value) of f(x) if and only if c is in the domain of f(x) and either $f'(c) = 0$ or $f'(c)$ is undefined. If c is a critical number of f(x), then the ordered pair $(c, f(c))$ is a critical point of f(x). In particular, since any polynomial is both defined and differentiable at x for each real number x, then c is a critical number of a polynomial p(x) if and only if $p'(c) = 0$.

The Product Rule for derivatives appears in various forms of generality. Most commonly, beginning calculus students are presented with the familiar formula for the derivative of the product of two differentiable functions which states that

$$\frac{d}{dx} f(x) \cdot g(x) = g(x) \cdot \frac{d}{dx} f(x) + f(x) \cdot \frac{d}{dx} g(x),$$

or

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

[5, p. 185]. Some texts extend the basic result for two functions to the formula

$$(fgh)'(x) = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$

[2, p. 145, no. 40] for the derivative of the product of three differentiable functions. Still others extend the basic result to the product of four differentiable functions, stating that

$$(fghk)'(x) = f'(x)g(x)h(x)k(x) + f(x)g'(x)h(x)k(x) + f(x)g(x)h'(x)k(x) + f(x)g(x)h(x)k'(x)$$

[3, p. 168, no. 40]. However, the Generalized Product Rule for Derivatives extends this fundamental principle to the arbitrary finite product of differentiable functions.

Generalized Product Rule for Derivatives: Suppose n is a positive integer, $f_k(x)$ is a differentiable function for $1 \le k \le n$, and $f(x) = \prod_{k=1}^{n} f_k(x)$. Then

$$f'(x) = \sum_{i=1}^{n} \left[f_i'(x) \cdot \prod_{k \ne i} f_k(x) \right].$$

In other words, if $f(x) = f_1(x) \cdot f_2(x) \cdot \cdots \cdot f_{n-1}(x) \cdot f_n(x)$, then

$$f'(x) = f_1'(x) \cdot f_2(x) \cdot \cdots \cdot f_n(x) + \cdots + f_1(x) \cdot \cdots \cdot f_{n-1}(x) \cdot f_n'(x)$$

[1, p. 187, Corollary 6.1.4].

Main Result

We are now prepared to present the main result of the paper. It is shown that if p(x) is a polynomial, $\deg(p(x)) = n \geq 3$, and p(x) has distinct real zeros $a_1 < a_2 < \cdots < a_n$, then p(x) has a complete set $\{c_i\}_{i=1}^{n-1}$ of critical numbers such that $a_i < c_i < a_{i+1}$ for $1 \leq i \leq n-1$. Furthermore, it is verified that the smallest critical number c_1 is nearer to the smallest zero a_1 than it is to the next larger zero a_2. Finally, a form of symmetry is established by also showing that the largest critical number c_{n-1} is nearer to the largest zero a_n than it is to the next smaller zero a_{n-1}.

Theorem: Suppose that p(x) is a polynomial of degree $n \geq 3$ containing n distinct real zeros $\{a_i\}_{i=1}^{n}$, where $a_i < a_{i+1}$ for $1 \leq i \leq n-1$.

(a) Then p(x) has a complete set $\{c_i\}_{i=1}^{n-1}$ of precisely $n-1$ critical numbers which have the property that $c_i \in (a_i, a_{i+1})$ for $1 \leq i \leq n-1$.

(b) Furthermore, c_1 is nearer to a_1 than a_2.

(c) Similarly, c_{n-1} is nearer to a_n than a_{n-1}.

Proof: Suppose p(x) is a polynomial, $\deg(p(x)) = n \geq 3$, p(x) has real zeros $\{a_i\}_{i=1}^{n}$, and $a_i < a_{i+1}$ for $1 \leq i \leq n-1$. Consequently, if p(x) has leading coefficient r, then $p(x) = r \cdot \prod_{k=1}^{n}(x - a_k)$, where $r \neq 0$.

(a) Since $\deg(p(x)) = n$ then $\deg(p'(x)) = n-1$. Furthermore, the critical numbers of p(x) are the zeros of $p'(x)$. Thus p(x) has at most $\deg(p'(x)) = n-1$ distinct critical numbers. However, since p(x) has distinct real zeros $\{a_i\}_{i=1}^{n}$ with the property that $a_i < a_{i+1}$ for $1 \leq i \leq n-1$, then by Rolle's Theorem p(x) has at least $n-1$ distinct critical numbers $\{c_i\}_{i=1}^{n-1}$, where $c_i \in (a_i, a_{i+1})$ for $1 \leq i \leq n-1$. Consequently $\{c_i\}_{i=1}^{n-1}$ is the complete set of critical numbers of p(x), where $c_i \in (a_i, a_{i+1})$ for $1 \leq i \leq n-1$. In particular, p(x) has critical numbers $c_1 \in (a_1, a_2)$ and $c_{n-1} \in (a_{n-1}, a_n)$.

(b) Define d_1 and d_2 to be the distances from c_1 to a_1 and a_2, respectively. Since $a_1 < c_1 < a_2$, then $d_1 = c_1 - a_1$ and $d_2 = a_2 - c_1$.

Now define $q(x) = \prod_{k=3}^{n}(x - a_k)$. Since $n = \deg(p(x)) \geq 3$, it follows that $\deg(q(x)) \geq 1$ and $p(x) = r(x - a_1)(x - a_2)q(x)$. Therefore

$$p'(x) = r[(x - a_2)q(x) + (x - a_1)q(x) + (x - a_1)(x - a_2)q'(x)]$$

by the Generalized Product Rule for Derivatives. Since c_1 is a critical number of p(x) then

$$0 = p'(c_1) =$$

$$r[(c_1 - a_2)q(c_1) + (c_1 - a_1)q(c_1) + (c_1 - a_1)(c_1 - a_2)q'(c_1)].$$

On the other hand, since $\{a_i\}_{i=1}^{n}$ is the complete set of zeros of p(x), $a_i < a_{i+1}$ for $1 \leq i \leq n-1$, and $a_1 < c_1 < a_2$, then $p(c_1) \neq 0$. Therefore

$$0 = \frac{p'(c_1)}{p(c_1)} =$$

$$\frac{r[(c_1 - a_2)q(c_1) + (c_1 - a_1)q(c_1) + (c_1 - a_1)(c_1 - a_2)q'(c_1)]}{r(c_1 - a_1)(c_1 - a_2)q(c_1)} =$$

$$\frac{1}{c_1 - a_1} + \frac{1}{c_1 - a_2} + \frac{q'(c_1)}{q(c_1)} =$$

$$\frac{1}{d_1} - \frac{1}{d_2} + \frac{q'(c_1)}{q(c_1)},$$

and so $\frac{1}{d_2} - \frac{1}{d_1} = \frac{q'(c_1)}{q(c_1)}$.

Note that $q(c_1) = \prod_{k=3}^{n}(c_1 - a_k)$. Applying the Generalized Product Rule for Derivatives again, we have $q'(c_1) = \sum_{i=3}^{n}\left[\prod_{\substack{k=3 \\ k\neq i}}^{n}(c_1 - a_k)\right]$. Therefore

$$\frac{1}{d_2} - \frac{1}{d_1} = \frac{q'(c_1)}{q(c_1)} = \frac{\sum_{i=3}^{n}\left[\prod_{\substack{k=3\\k\neq i}}^{n}(c_1 - a_k)\right]}{\prod_{k=3}^{n}(c_1 - a_k)} =$$

$$\frac{\prod_{\substack{k=3\\k\neq 3}}^{n}(c_1 - a_k)}{\prod_{k=3}^{n}(c_1 - a_k)} + \frac{\prod_{\substack{k=3\\k\neq 4}}^{n}(c_1 - a_k)}{\prod_{k=3}^{n}(c_1 - a_k)} + \cdots + \frac{\prod_{\substack{k=3\\k\neq n}}^{n}(c_1 - a_k)}{\prod_{k=3}^{n}(c_1 - a_k)} =$$

$$\frac{1}{c_1 - a_3} + \frac{1}{c_1 - a_4} + \cdots + \frac{1}{c_1 - a_n} =$$

$$\sum_{i=3}^{n}\left[\frac{1}{c_1 - a_i}\right].$$

Since $c_1 \in (a_1, a_2)$ and $a_i < a_{i+1}$ for $1 \le i \le n-1$, it follows that $c_1 < a_i$ for $3 \le i \le n$, and so $\frac{1}{c_1 - a_i} < 0$ for $3 \le i \le n$. Thus $\frac{1}{d_2} - \frac{1}{d_1} = \sum_{i=3}^{n}\left[\frac{1}{c_1 - a_i}\right] < 0$, so that $d_1 < d_2$. Hence c_1 is nearer to a_1 than a_2.

(c) Following a strategy similar to that of part (b), we define d_3 and d_4 to be the distances from c_{n-1} to a_{n-1} and a_n, respectively. Since $a_{n-1} < c_{n-1} < a_n$, then $d_3 = c_{n-1} - a_{n-1}$ and $d_4 = a_n - c_{n-1}$.

Now define $Q(x) = \prod_{k=1}^{n-2}(x - a_k)$. Since $n = \deg(p(x)) \ge 3$, it follows that $\deg(Q(x)) \ge 1$ and $p(x) = rQ(x)(x - a_{n-1})(x - a_n)$. Therefore

$$p'(x) = r[Q'(x)(x - a_{n-1})(x - a_n) + Q(x)(x - a_n) + Q(x)(x - a_{n-1})]$$

by the Generalized Product Rule for Derivatives. Since c_{n-1} is a critical number of p(x) then

$$0 = p'(c_{n-1}) =$$

$$r[Q'(c_{n-1})(c_{n-1} - a_{n-1})(c_{n-1} - a_n) + Q(c_{n-1})(c_{n-1} - a_n) + Q(c_{n-1})(c_{n-1} - a_{n-1})].$$

On the other hand, since $\{a_i\}_{i=1}^{n}$ is the complete set of zeros of p(x), $a_i < a_{i+1}$ for $1 \le i \le n-1$, and $a_{n-1} < c_{n-1} < a_n$, then $p(c_{n-1}) \neq 0$. Therefore

$$0 = \frac{p'(c_{n-1})}{p(c_{n-1})} =$$

$$\frac{r[Q'(c_{n-1})(c_{n-1}-a_{n-1})(c_{n-1}-a_n)+Q(c_{n-1})(c_{n-1}-a_n)+Q(c_{n-1})(c_{n-1}-a_{n-1})]}{rQ(c_{n-1})(c_{n-1}-a_{n-1})(c_{n-1}-a_n)} =$$

$$\frac{Q'(c_{n-1})}{Q(c_{n-1})} + \frac{1}{c_{n-1}-a_{n-1}} + \frac{1}{c_{n-1}-a_n} =$$

$$\frac{Q'(c_{n-1})}{Q(c_{n-1})} + \frac{1}{d_3} - \frac{1}{d_4},$$

and so $\frac{1}{d_4} - \frac{1}{d_3} = \frac{Q'(c_{n-1})}{Q(c_{n-1})}$.

Note that $Q(c_{n-1}) = \prod_{k=1}^{n-2}(c_{n-1}-a_k)$. Applying the Generalized Product Rule for Derivatives again, we have $Q'(c_{n-1}) = \sum_{i=1}^{n-2}\left[\prod_{\substack{k=1\\k\neq i}}^{n-2}(c_{n-1}-a_k)\right]$. Therefore

$$\frac{1}{d_4} - \frac{1}{d_3} = \frac{Q'(c_{n-1})}{Q(c_{n-1})} = \frac{\sum_{i=1}^{n-2}\left[\prod_{\substack{k=1\\k\neq i}}^{n-2}(c_{n-1}-a_k)\right]}{\prod_{k=1}^{n-2}(c_{n-1}-a_k)} =$$

$$\frac{\prod_{\substack{k=1\\k\neq 1}}^{n-2}(c_{n-1}-a_k)}{\prod_{k=1}^{n-2}(c_{n-1}-a_k)} + \frac{\prod_{\substack{k=1\\k\neq 2}}^{n-2}(c_{n-1}-a_k)}{\prod_{k=1}^{n-2}(c_{n-1}-a_k)} + \cdots + \frac{\prod_{\substack{k=1\\k\neq n-2}}^{n-2}(c_{n-1}-a_k)}{\prod_{k=1}^{n-2}(c_{n-1}-a_k)} =$$

$$\frac{1}{c_{n-1}-a_1} + \frac{1}{c_{n-1}-a_2} + \cdots + \frac{1}{c_{n-1}-a_{n-2}} =$$

$$\sum_{i=1}^{n-2}\left[\frac{1}{c_{n-1}-a_i}\right].$$

Similar to part (b), since $c_{n-1} \in (a_{n-1}, a_n)$ and $a_i < a_{i+1}$ for $1 \le i \le n-1$, then $c_{n-1} > a_i$ for $1 \le i \le n-2$, and so $\frac{1}{c_{n-1}-a_i} > 0$ for $1 \le i \le n-2$. Therefore $\frac{1}{d_4} - \frac{1}{d_3} = \sum_{i=1}^{n-2}\left[\frac{1}{c_{n-1}-a_i}\right] > 0$, so that $d_3 > d_4$. Hence c_{n-1} is nearer to a_n than a_{n-1}.

Concluding Remarks

The main theorem generalizes the problem posed by Moran [4] in two ways. By specifying that $\deg(p(x)) \ge 3$, the problem posed by Moran for cubic polynomials is simply the special case in which $\deg(p(x)) = 3$. Furthermore, in addition to showing that the smallest critical number c_1 of $p(x)$ is closer to the smallest zero a_1 than the next to smallest zero a_2, a symmetrical relationship is established by showing that the largest critical number c_{n-1} of $p(x)$ is closer to the largest zero a_n than the next to largest zero a_{n-1}. Finally, even though Waterhouse provided the solution for the critical number $c_1 \in (a_1, a_2)$ in the more general case in which $\deg(p(x)) \ge 3$ [6, pp. 345-346, Problem 459], his solution did not include the symmetrical result for $c_{n-1} \in (a_{n-1}, a_n)$.

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References

1. Bartle, Robert G. and Sherbert, Donald R., *Introduction to Real Analysis*, second edition, John Wiley and Sons, Inc., New York, New York, 1992.
2. Grossman, Stanley, *Calculus*, fifth edition, Harcourt Brace and Company, Orlando, Florida, 1992.
3. Mizrahi, Abe and Sullivan, Michael, *Calculus and Analytic Geometry*, third edition, Wadsworth Publishing Company, Belmont, California, 1990.
4. Moran, D. A., *Problem 459*, The College Mathematics Journal, Vol. 22, No. 4, (1991) p. 344.
5. Stewart, James, *Single Variable Calculus: Early Transcendentals*, seventh edition, Brooks/Cole, Belmont, California, 2012.
6. Waterhouse, William C., *Problem 459*, The College Mathematics Journal, Vol. 23, No. 4, (1992) pp. 345-346.
7. Williams, Rob, *Problem 459*, The College Mathematics Journal, Vol. 23, No. 4, (1992) p. 345.