On the Hahn-Banach Theorem and Its Application to Polynomial and Dirichlet Problems

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Abstract

Extension posed a serious mathematical problem in functional analysis until Hahn and Banach came up with the efficient Hahn-Banach theorem for extension of functionals in Banach space. Subsequently other extension results have continued to evolve even in topology and functional analysis. In section two of this work, the basics of these results were considered precisely as theorems and propositions. These strongly gave rise to the background of the applications considered in this work as in section three.

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1. INTRODUCTION

Knowing that a linear function on a vector space X is a linear operator from X to the space $\Re$ of real numbers and that a linear functional is a real valued function on X such that $f(\alpha x + by) = \alpha f(x) + bf(y)$, we then ask a first question on how we can extend a linear operator (or functional) from such a space to the whole space X in such a way that the various properties of the functional are preserved. In view of this, a fundamental extension theorem, the Hahn- Banach theorem and its consequent corollaries preliminarily came into play extensively in section two, the next section.

As second question, we assume that the function g is given for only a subset X of the real space, C, can we extend X so as to define a linear functional g of norm $M_g \leq M$ in the entire space C. In view of this, an evident necessary condition is that

for all linear combinations of elements of X, we have

$$\left| \sum_{i=1}^{n} ck \;\; gf_k \right| \leq M \left| \sum_{i=1}^{n} ckf_k \right| \qquad \ldots \qquad (1.1)$$

If we let M_g to be the smallest value of M for which (1.1) is fulfilled, we observe that condition (1.1) becomes a sufficient condition for answer to the above immediate question.

2. PRELIMINARY RESULTS ON EXTENSION

Theorem 2.1: (Hahn-Banach): let p be a real- valued function defined a vector space X satisfying $p(x+y) \leq p(x) + p(y)$ and $p(\alpha x) = \alpha p(x)$ for each $\alpha \geq 0$. Suppose that f is a linear functional defined on a subspace S and that $f(s) \leq p(s)$ for all s in S. then there is a linear functional f defined on X such that $f(x) \leq p(x)$ for all x, and $f(s) = f(s)$ for all s in S.

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Proof: Consider all liner functional g defined on a subspace of X and satisfying $g(x) \leq p(x)$ Whenever g(x) is defined. This set is partially ordered by setting $g_1 < g_2$ if g_2 is an extension of g_1, that is, if the domain of $g_1$is contained in the domain of g_2 and $g_1 = g_2$ of the domain of g_1.

By the Hausdorff Maximal Principle there is a maximal linearly ordered subfamily $\{g_\alpha\}$ that contains the given functional f. We define a functional which is independent of the α chosen. The domain of f is a subspace and f is a linear functional, for if x and y are in the domain of f. then $x\in$ domain g_α and $y \in$ domain g_β for some α, β. By the linear ordering of $\{g_\alpha\}$, we have either $g_\alpha< g_\beta$ or $g_\beta < g_\alpha$, say the former. Then x and y are in the domain of g_β, and so $\lambda x + \mu y$ is in the domain of g_β and so in the domain of f, and $f(\lambda x+\mu y) = g_\beta(\lambda x+\mu y) =\lambda g_\beta(x) + \mu g_\beta(y) = \lambda f(x) + \mu f(y)$. Thus f is an extension of f. Moreover, f is a maximal extension for if G is any extension of f, $g_\alpha < f< G$ implies that G must belong to (g_α) by the maximality of (g_α). Hence $G < f$, and so $G = f$.

It remains only to show that f is defined for all $x \in X$. since f is maximal. This will follow if we can show that each g that is defined on a proper subspace T of X and satisfies $g(t) \leq p(t)$ has a proper extension h.

Let y be an element in X ~T. we shall show that g may be extended to the subspace U spanned by T and y, that is, to the subspace consisting of elements of the form $\lambda y + t$ with $t \in T$. if h is an extension of g, we must have

$h(\lambda y + t) = \lambda h(y) + h(t) = \lambda h(y) + g(t)$,

and so h is defined as soon as we specify h(y).

for $t_1, t_2, \in T$ we have

$g(t_1) + g(t_2) = g(t_1 + t_2) \leq p(t_1+t_2) \leq p(t_1-y) + p(t_2-y)$. Hence

$-p(t_1-y) + g(t_1) \leq p(t_2+y)-g(t_2)$, and so

$$\sup_{t \varepsilon T}[-p(t-y)+g(t)] \leq \inf_{t \varepsilon T}[p(t+y)-g(t)].$$

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Define $h(y) = \alpha$, where α is a real number such that $\sup\{-p(t-y) + g(t)\} \leq \alpha \leq \inf [p(t+y)-g(t)$. we must show that $h(\lambda y+t) = \lambda\alpha+g(t) \leq p(\lambda y + t)$.

if $\lambda > 0$, then $\lambda\alpha + g(t) = \lambda\{\alpha + g(t/\lambda)\}. \leq \lambda = \lambda [\{p(t/\lambda + y)-g(t/\lambda)\} + g(t/\lambda)]$

$= \lambda p(t/\lambda+ y) = p(t + \lambda y)$. if $\lambda = - \mu < 0$,

then $\alpha \mu + g(t) = \mu(- \alpha + g(t/\mu)) \leq \mu (\{p(t/\mu - y) - g(t/\mu)\} + g(t/\mu))$

$= \mu p(t/\mu -y) = p(t = \mu y)$.

Thus $h(\lambda y + t) \leq p(\lambda y + t)$ for all λ, and h is a proper extension of g.

The following proposition is a generalization of the Hahn- Banach Theorem which is useful in certain applications. By an Abelian semigroup of linear operators on a vector space X, we mean a collection G of linear operators from X to X such that of A and B are in G, then AB = BA and AB is in G. we also assume that the identity operator belongs to G.

Proposition 2.1: let x, s, p, and f be as in Theorem 2.1, and let G be all Abelian semigroup of linear operators on X such that for every A in G we have $p(Ax) \leq p(x)$ for all x in X, while for each s in S we have As in S and f(As) = f(s). then there is an extension f of F to a linear functional on X such that $f(x) \leq p(x)$ and F(Ax) for all x in X [10]

Proof: define a function q on X by setting

$q(x) = \inf \dfrac{1}{nm} p\,(A_1x + \ldots + A_nx)$,

Where the inf is taken over all finite sequences $\{A_1,\ldots A_n\}$ from G.

we clearly have $q(x) \le p(x)$ and $q(\alpha x) = \alpha q(x)$ for $\alpha \ge 0$. for any x and y in X and any pairs $\{A_1,\ldots A_n\}$ and $\{B_1,\ldots B_m\}$ of finite sequences from G we have

$$q(x+y) \le \frac{1}{nm} p\left(\sum_{i=1}^{n} \sum_{i=1}^{m} A_i B_j (x+y)\right)$$

$$\le \frac{1}{nm} p\left(\sum_{j=1}^{m} B_j\left(\sum_{i=1}^{n} A_i x\right)\right) + \frac{1}{nm} p\left(\sum_{i=1}^{n} A_i\left(\sum_{j=1}^{m} B_j y\right)\right)$$

$$\le \frac{1}{n} p\left(\sum_{i=1}^{n} A_i x\right) + \frac{1}{m} p\left(\sum_{j=1}^{m} B_j y\right).$$

Taking infima over every pair $\{A_i\}$, $\{B_j\}$, we obtain

$q(x + y) \le q(x) + q(y)$.

Since $q(\theta) = p(\theta) = 0$, we have

$0 = q(x-x) \le q(x) + q(-x) \quad \le q(x) + p(-x)$

Thus q(x) cannot be $-\infty$, and q is real valued.

For s in S,

$$f(s) = \frac{1}{n} f(A_1 s + \ldots + A_n s) \le \frac{1}{n} p(A_1 s + \ldots + A_n s)$$

Hence $f(s) \le q(s)$, and we may apply Theorem 2.1 with p replaced by q to obtain an extension f of F to all z such that $f(x) \le q(x) \le p(x)$. it remains only to show that $f(Ax) = f(x)$. Now $q(x-Ax) \le$

$$\frac{1}{n} p((x - Ax) + A(x - Ax) + \ldots + A^n (x - Ax))$$

$$= \frac{1}{n} p(x - A^{n+1} x) \le \frac{1}{n}[p(x) + p(-x)].$$

Since this is true for each n, we have $q(x-Ax) \le 0$. since $f(x) - f(Ax) = f(x-Ax) \le q(x-Ax) \le 0$,

we have f(Ax), and applying this to –x, we get $f(x) = f(Ax)$.

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Proposition 2.2: let x be an element in a normed vector space X. Then there is a bounded linear functional f on X such that $f(x) = \|f\| \, \|x\|$. [3]

Proof: let S be the subspace consisting of all multiples of x, and define f on S by $f(\lambda x) = \lambda \| x \|$ and set $p(y) = \| y\|$. Then by the Hahn – Banach Theorem there is an extension of f to be a linear functional on X such that $f(y) \le \|y\|$. Since $f(-y) \le \|y|$, we have $\|f\| \le 1$. Also $f(x) = \|x\| \le \|f\|. \|x\|$. Thus $\|f\| =1$ and $f(x) = \|f\|. \| x\|$.

Proposition 2.3: let T be a linear subspace of a normed linear space X and Y Element of X whose distance to T is at least δ, that is, and element such that $\| y -t\| \ge \delta$ for all $t \,\varepsilon\, T$. then there is a bounded linear functional f on x with $\|f\| \le 1$, $f(y) = \delta$, and such that $f(t) = 0$ for all t in T. [5]

Proof: let S be the subspace spanned by T and y, that is, the subspace consisting of all elements of the form $\alpha y + t$ with $t \,\varepsilon\, T$.
Define $f(\alpha y + t) = \alpha\delta$, we have $f(s) \le \| s\|$ on S. by the Hahn – Banach Theorem we may extend f to all of X so that $f(x) \le \|x\|$. But this implies that $\|f\| \le 1$. By the definition of f on S, we have $f(t) = 0$ for $t \in T$ and $f(y) = \delta$.

The space of bounded linear functional on a normed space X is called the dual (or conjugate) of X and is denoted by X^*. Since R is complete, the dual X^* of any normed space X is a Banach space by Proposition 2.3. Two normed vector spaces are said to be isometrically isomorphic if there is a one to one linear mapping of one of them into the other which preserves norms. From an abstract point of view, isometically isomorphic spaces are identical, the isomorphism merely amounting to a refinent of the elements. I suppose we know that the dual of L^P was (isometrically isomorphic to) L_q for $1\le p < \infty$ and that there was a natural representation of the bounded linear functional on L_p by elements of L_q.

We are now in a position to show that a similar representation does not hold for bounded linear functional on $L^\infty[0,1]$. We note that C[0,1] is a closed

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subspace of $L^{\infty}[0,1]$. Let f be that linear functional on C[0,1] which assigns to each x in C[0,1} the value x(0) of X at 0. It has norm 1 on C, and so can be extended to a bounded linear functional f on $L^{\infty}[0,1]$.

Linear functional and Hahn-Banach Theorem

Given $L^1[0,1]$ such that f(x) = $\int_0^1 xy$ dt for all x in C, let $\{x_n\}$ be a sequence of continuous functions on [0,1] that are bounded by 1, we have $x_n(0) = 1$, and such that $x_n(t) \to 0$ for all $t \neq 0$. Then, for each

$$y \in L^1, \int x_n \, dxy \to 0, \ \ while \ \ f(x_n) = 1.$$

If we consider the dual x** of x*, then to each x in X, there corresponds an element φx in x** defined by (φx)(f) = f(x). We have ||x|| = sup f(x). Since f(x) ≤ ||f|| ||x||, we have ||φx|| ≤ ||x||, ||f|| =1.

While by proposition 2.2 we have an f of norm 1 with f(x) = ||x||.

Hence ||φx|| = ||x||. Since φ is clearly a linear mapping, φ is an isometric isomorphism of x onto some linear subspace φ [x] of x**. The mapping φ is called the natural isomorphism of x onto x**, and if φ[x] = x** we say that x is reflexive.

Thus L_p is reflexive if $1 < p < \infty$. Since there are functionals on L^{∞} that are not given by integration with respect to a function on ℓ_1 L^1, it follows that L^1 is not reflexive. It should be observed that x may be isometric with x** without being reflexive.

By Proposition 2.3, the space x** is complete, and so the closure φ[x] vector space is isometrically isomorphic to a dense subset of a Banach space.

Before closing this section, we add a word about the Hahn –Banach Theorem for complex vector spaces. A complex vector space is a vector space in which we allow multiplication by complex scalars. The following extension of the Hahn-Banach Theorem for complex spaces is due to Bohnenblust and Sobczyk:

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Theorem 2.2: let X be a complex vector space, S a linear subspace, p a real –valued function on X such that $p(x+y) \leq p(x) + p(y)$, and $p(\alpha x) = |\alpha| p(x)$. let f be a (complex) linear functional on S such that $|f(s)| \leq p(s)$ for all s in S. Then there is a linear functional f defined on x such that f(s) = f(s) for s in S and $|f(x)| \leq p(x)$ for all x in X. [6]

Proof: we first note that x can be considered as a real vector space if we simply ignore the possibility of multiplying by complex constants. A mapping f from X to the complex numbers that is linear in the real sense is linear in the complex sense if an only if f(ix) = if(x) for each x. on S define g and h by taking g(s) to be the real part of f(s) and h(s) the imaginary part. Then g and h are linear in the real sense and f = f + ih. Since f is linear in the complex sense, g(is) + ih(is) = f(s) = ig(s) – h(s), and so h(s) = - g(is).

Since $g(s) \leq |f(s)| \leq p(s)$, we can extend g to a functional g on X that is linear in the real sense and satisfies $G(x) \leq p(x)$. let F(x) = G(x) – iG(ix). Then F(s) for s in S. since F(ix) = G(ix) –iG(ix) = i[G(x) – iG(ix)], we have F linear in the complex sense. For any X, choose ω to be a complex umber such that |ω| = 1 and ωF(x) = |F(x)|. Then $|F(x)| = \omega F(x) = F(\omega x) = G(\omega x) \leq p(\omega x) = p(x)$.

3. APPLICATIONS

3. 1. THE DIRICHLET PROBLEM

If Ω be a bounded domain in R^n and let $\Delta = \sum_{j=1}^{n} \partial^2 / \partial x_j^2$..the equation $\Delta u = \partial s$ called the Laplace equation. Any function in $C^2(\Omega)$ that satisfies the Laplace equation in Ω is called a harmonic function in Ω. We shall denote by $\partial\Omega$ the boundary of Ω. We say that $\partial\Omega$ is in C^2 if $\partial\Omega$ can be covered by a finite number of open subsets G_i, with each $G_i \cap \partial\Omega$ having a parametric representation in terms of functions in C^2. The Dirichlet is the following one: given a continuous function f on $\partial\Omega$ continuous in $\bar{\Omega}$, and satisfying

$$\Delta u = 0 \quad in\ \Omega, \qquad \ldots \qquad (3.1.1)$$

$$u = f \qquad \ldots \qquad (3.1.2)$$

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Theorem 3.1.1 The Dirichlet problem has at most one solution.

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This follows immediately from the following result, known as the (weak) maximum principle.

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Theorem 3.1.2 Let u be a harmonic function in Ω, continuous in $\bar{\Omega}$.
Then $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$ [2]

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Proof. If the assertion is not true, then there is a point $y \in \Omega$ such that $u(y) > \max_{\partial\Omega} u$. consider the function $v(x) = u(x) + \varepsilon\,|\,x - y\,|^2$ if $.\varepsilon > 0$ is sufficiently small, then $v(y) = u(y) > \max_{\partial\Omega} v$ is attained at a point z in Ω. At the point,

$$\frac{\partial v}{\partial x_i} = 0, \frac{\partial^2 v}{\partial x_1^2} \le 0 \qquad (i = 1, \ldots n)$$

Hence $\Delta v \le 0$ at z .however,

$\Delta v = \Delta u + \varepsilon\Delta\left(|\,z - y\,|^2\right) = 2n\varepsilon > 0$ a contradiction as for the existence of a solution to the Dirichlet problem, we quote the following result:

Theorem 3.1.3 If $\partial\Omega$ is in C^2, then for any continuous function f on $\partial\Omega$ there exists a solution to the Dirichlet problem (3.1.1), 3.1.2). [2]

Theorem 3.1..3 Is much deeper than Theorem 3.1.1. There are various methods of providing it, but each of them requires lengthy developments. Consider now the function

$$k(x,y)=\begin{cases} |x-y|^{2-n}, & if\ n>3, \\ -\log|x-y|, & if\ \ n=2. \end{cases}$$

We call it a fundamental solution of the Laplace equation. It satisfies the Laplace equation in x, when x varies in $\Omega-\{y\}$, and it grows to ∞ as $x\to y$. [7]

Definition 3.1.1 A function $G(x,y)$, defined for each $y\in\Omega\ and\ \ x\in\bar{\Omega}-(y)$, is called Green's function (for the Laplace operator in Ω) if:

(i) $G(x,y)=k(x,y)+h(x,y), where\ h(x,y)$ is harmonic in x when $x\in\Omega$

(ii) $G(x,y)$ is continuous in x when $x\in\overline{\Omega}-[y]$.

(iii) $G(x,y)=0\,for\ \ x\in\partial\Omega.$ [2]

One can use Green's function in order to represent the solution u of the Dirichlet problem (3.1.1), (3.1.2) in terms of an integral involving the boundary values f . What we shall now do is construct Green's function by using, as a tool, the Hahn-Banach theorem. We shall consider only the case n = 2, which is somewhat simpler than the case where $n\geq 3$. we shall assume that $\partial\Omega$ consists of a definite number of continuously differentiable closed curves. The following condition then holds (the proof is omitted):

(P) For any z near $\partial\Omega$, denote by $T_{=}$ the tangent plane to $\partial\Omega$ at the point on $\partial\Omega$ nearest to z. denote by z' the reflection of z with respect to $T_{=}$

then

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$$\max_{x\in\partial\Omega}\frac{|z-x|}{|z'-x|}\to 1 \quad if \quad z\to z_0, z_0\in\partial\Omega$$

Theorem 3.1.4 if $n=2$ and $\partial\Omega$ is in C^1, then Green's function exists.

Proof. Denote by x the Banach space of all continuous functions on $\partial\Omega$ with the maximum norm, and denote by x' the linear subspace consisting of those functions f for which the Dirichlet problem (3.1.1), (3.1.2) has a solution. For an $y\in\Omega$, consider the linear functional L_y on X' defined by $L_y(f) = u(y)$, where u is the solution of (3.1.1), (3.1.2). From Theorem 3.1.2 it follows that L_y is bounded and that its norm is 1.

By the Hahn- Banach theorem L_y can be extended into a bounded linear functional on X, having norm 1. We denote such an extension again by L_y.

For each $z\notin\partial\Omega$ consider the element $f_z(x)=\log|x-z| \quad (x\in\partial\Omega)$ and define $k_y(z)=L_y(f_z)$. We claim that $k_y(z)$ is a harmonic function. To prove it let $z'=(z_1+\delta, z_2)$. *then*

$$\frac{k_y(z')-k_y(z)}{\delta}=L_y\left(\frac{f_z-f_z}{\delta}\right)$$

As $\delta\to 0,(f_z\to f_z)/\delta\to\partial f_z/\partial z_1, \quad where \quad \partial f_z/\partial z_1$ is the elements $\partial(\log|x-x|)/\partial z_1$ of x. since L_y is a continuous operator on X, we get

$$\frac{\partial k_y(z)}{\partial z_1} \text{ exists and equals } L_y\left(\frac{\partial fz}{\partial z_1}\right)$$

3.2. SPACES OF POLYNOMIALS

3.2.1 Introduction

We know that ordinarily it is impossible to extend scalar valued continuous homogenous polynomials over a larger Banach space using the popular Hahn-

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Banach theorem unless by the use of a more rigorous but a stronger approach which involves to an extent the use of an ultra power method which can be related to the existence of such extension morphism by complementing X** in G** in f: G →X* when a = X**. This is just exactly the same as the existence of a linear extension morphism for continuous homogenous polynomial $p(^kF) \to p(^kG)$ and our interest is the perfect Hahn-Banach type extension with sufficient condition for the existence of each p and not a linear morphism that will extend all of them. We do this by identifying polynomials more so, we know that the space $p(^kX)$ of K- homogenous polynomials over a Banach space X is the dual space of the space of symmetric kitensor of elements of X with the projective topology $p(^kX)$ being the dual of a space spanned by evaluations.

Normed with linear functional suit that we use Hahn- Banach theorem to extend them.

We consider the following before we deliver into the main application. Given an element $x \in X$, at x^i is a continuous linear form over the space of K-homogenous polynomials we denote the evaluation morphism by e. The norm of e_x as an element of $p(^kX)$ ' is $\|x\|^k$. denoted by S the linear subspace of $p(^kX)$ ' spanned by all evaluations at points of X. This is a (non-closed) subspace whose elements have non- unique representations of the form $s = \sum_{j=1}^{n} e_{xj}$,

For complex X the norm of such an element is

$$\|s\| = \sup\left\{ \left| \sum_{j=1}^{n} P(xj) \right| : P \in P(^k X) \ \text{of norm one} \right\}$$

which is independent of the representation of s. When x is a real Banach space and k is even, the representation must take into account the signs, so s = $\sum_{j-1}^{n} e_{xj} - \sum_{i=1}^{m} e_{yi.}$ as our results are valid for both the real and complex case, but we will use only the complex-case notation. Note also that for any scalar λ, $e_{\lambda x} = \lambda^k e_x$. The dual of s can be readily seen to be $p(^kX)$.

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We will need to consider other topologies on the vector space s. we know from |14| that the Borel transform B: $p(^kX)' \rightarrow p(^kX')$ is the linear map given by B(T) (γ^k). A linear form T is the kernel of the B if and only if T is zero over the space of approximable polynomials. Since generally not all polynomials are approximable, B is rarely one to one. However, the following lemma assures that when restricted to the space S, the Borel transform is always one-to-one.

Lemma[3.2.1]: let$x_1,\ldots,x_n, \in x$ if

$$\sum_{j=1}^{n} \gamma(x_j)^k = 0 \quad for \quad all\, \gamma \in x', then \sum_{j=1}^{n} p(x_j) = 0 \; for all\, P \in P(^k x).$$

$p(^kX)'$.

Proposition 3.2.1: S' is isometrically isomorphic to $P_1(^kX)$.

Note that an immediate consequence of the proposition is that the image of the Borel Map p: $P(^kx)' \rightarrow P(^kx')$ is the space of integral k-homogeneous polynomials over

x'. Although we are mainly concerned with integral polynomials, we will take the

mapping $T \leftrightarrow P_T$ where $P_T(x)=T(e_x)$ (as in theorem), is an isomorphism between the algebraic dual of S and the space of all (not necessarily continuous) k-homogeneous polynomials over X (we will denote its reverse $P \rightarrow T_p$).

Imposing more or less stringent continuity conditions on T will produce different kinds of polynomials. Hence we show that all spaces of polynomials are produced in this way.

Proposition 3.2:1 If Z is any subspace of $P(^kx)$ containing the finite type polynomials, Z is (algebraically) isomorphic to $(s,\tau)'$, where $_T$ is a Hausdorff convex topology on S.

1. Extension Of Polynomials

In this section, we consider the problem of extending a polynomial defined on a Banach space X to a larger Banach space Y. thus, we will use the "linearization of different types of polynomials presented in section 3.1 in conjunction with the Hahn-Banach extension theorem for locally convex, to produce extensions of

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polynomials. We will identify in this section the linear functional $_{TP}$ with the polynomial P. thus we may write P∈ (s,τ)'.

Denote with S_X and S_Y the spaces spanned, as in the proceeding section, by evaluations in points of X and of Y. respectively. Then the inclusion map τ from X to Y induces a map from S_X to S_Y.

$$\sum_{j=1}^{n} e_{xj} \rightarrow \sum_{j=1}^{n} e_{\tau(xj)},$$

Which is one-to-one, thanks to the Lemma in section 3.1, and the Hahn-Banach theorem. Indeed, if $\sum_{j=1}^{n} e_{\tau(x_j)} = 0$ in S_X, and γ ∈ X', then extend γ to Γ ε Y', and we have

$$\left(\sum_{j=1}^{n} e_{xj}\right)(\gamma) = \left(\sum_{j=1}^{n} e_{\tau(x_j)}\right)(\Gamma) = 0 \text{, for any } \gamma \in X'.$$

Thus $\sum_{j=1}^{n} e_{xj} = 0.$ we will drop the τ in the sequel and consider any s ∈ S_X an element of S_Y.

Note that any P ∈ $(S_X)^*$ the algebraic dual of S_X extends to S_Y, so any polynomial can be algebraically extended; the real question is what type of polynomial we can expect this extension to be. Now consider locally convex, Hausdorff topologies τ_x and τy on S_X and S_y. Then the following proposition is immediate.

Proposition 3.2.1: given the spaces $(S_{x,}\ \tau_y)$' and $(S_y\ \tau_y)$' of polynomial over X and Y respectively, then a polynomial P ∈ $(S_{X,}\ \tau_x)$' extends to a polynomials $(S_{y,}$ $\tau_y)$' if it is τ_G-continuous (i.e. for the topology induced by τ_y on S_x). [12]

We now extend integral; polynomials over E to integral polynomials over an arbitrary larger space X.

***Theorem*:** Any P ∈ $P_I(^kE)$ can be extended to $\breve{P}$ ∈ $P_I(^kG)$, with $\|\breve{P}\|_I = \|P\|_I$ [14].

Corollary 3.2.1: There is an extendible, non- integral polynomial over $c_{o.}$ [12]

To see this, consider a non- weakly compact symmetric operator T: $\ell_1 \rightarrow l_\infty$, and consider the inclusion $\ell_1 \subset C([0,1])$. T corresponds to a 2-homogeneous polynomial P over ℓ_1, and its extension to C([0,1]) would give rise to a continuous symmetric operator S: C ([0,1]) $\rightarrow$ C ([0,1]) making the following diagram commutative.

$$\begin{array}{ccc} \ell_1 & \xrightarrow{T} & \ell_\infty \\ J \downarrow & & \uparrow J' \\ C([0,1]) & \xrightarrow{s} & C([0,1])' \end{array}$$

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But this can not be, for the symmetric regularity of C ([0,1]) assures the weak compactness of S, and thus of T. [16]

Many sub-classes (finite-type, nuclear) of integral polynomials over E can of course be extended to the corresponding kind of polynomial over G. we next state this kind of correspondence for polynomials in (S_E,τ_0)'.

Proposition 3.2.1: each P $\in$ (S_E,τ_0)' extends to $\overline{P} \in (S_G,\tau_0)$; [13].

Applying these to the integral polynomial, we know that according to Aron and Berner, any continuous homogenous polynomial can be extended from X to X' which indicates that extension is in fact purely algebraies in nature and cannot be applied to any polynomial contributions hence we verify that P is nuclear polynomial over X. then its Aron-Berner extension P is also nuclear. In fact, if P = $\Sigma_I \gamma$, then P = Σ_I with $\gamma = j(\gamma)$ where j X' $\rightarrow$X' is the canonical inclusion if P is integral and μ is a regular Borel on B_E representing P. then one is tempted to put for $\lambda \in$ X'.

$$\overline{P}(z) = \int_{B_X} z(\gamma)^k \, d\mu(\gamma)$$

where as the only problem with this expression is that this integral may not exist. In fact z^k need not be a μ- measurable function. Note that z^k is not a continuous function on (B_E'w*), nor is the point wise limit of a sequence of powers of elements of E (which are known to be integrable). [1]

We will prove that the validity of the above expression for the Aron-Berner extension of an integral polynomial is equivalent to E not containing an isomorphic copy of l_1. for this we will use the equivalence given by Haydon in [9], and the characterization of the Aron- Berner extension in [1]. We will also be able to prove that the Aron-Berner extension of an integral polynomial is always integral, with the same integral norm even when the above expression is not valid. μ will denote a regular Borel measures on (B_x,ω'), and k ∈ a positive integer. Define S, ℓ_1 (μ)→X' as S(f)(x) = $\int_{B_E} f(\gamma)\gamma(x)d\mu(\gamma)$ and consider S': X"→ ℓ_1 (μ)' = ℓ_∞ (μ). It easily follows that ||s'|| = ||s|| ≤ 1 and, for x ε X, s' (x) = x, where x is the class of the function γ→γ(x).

***Lemma 3.2.1*:** If P is an integral polynomial over X, represented by the measureμ, then its Aron-Berner extension P may be written

$$\overline{P}(z) = \int_{B_{E'}} S'(z)^k d\mu \text{ [14]}$$

***Corollary 3.2.2*:** The Aron- Berner extension of an integral polynomial is an integral polynomial and has the same integral norm. [7]

***Proof*:** A polynomial Q on a Banach space x is integral if and only if there exists a finite measure v on a compact space Ω and a bounded linear operator R: x → ℓ_∞ (Ω) such that

Q(x) =$\int_\Omega R(x)^k dv$

(see[10]). Moreover, we have $\|Q\|_I \leq \|R\|^k |v|$.

in our case, the previous lemma gives such a factorization and, since ||s'|| ≤ 1,$\|\overline{P}\|_I \leq \|P\|_I$. the other inequality is a consequence of our first proposition of ℓ_1, since p is an extension of as a linear functional.

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