Linear Algebra analyzing on Lights Out Game

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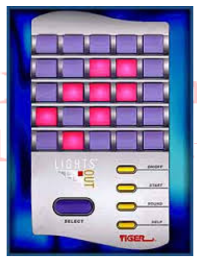
Abstract

In this paper, we analyze the linear equation system behind the Lights Out game. We have discussed how to do the solvable initial setting and how to get the minimal play solution. The main analyzing tool is MATLAB. We also design a window application for this game based on the above math analyzing.

§1. Introduction

Lights Out is an electronic game, released by Tiger Toys in 1995.



The game consists of a 5 by 5 grid of lights (buttons). When the game starts, a random number or a stored pattern of these lights is switched on. Pressing any of the lights will toggle it and the four adjacent lights. The goal of the game is to switch all the lights off, preferably in as few button presses as possible.

The basic topics in this paper are

(1) For what kind of initial lights setting, the game has solution?
(2) If it has solution, how to find the solution with minimal plays?
(3) How to design a window software application for this game?

Notes:

(1) Two consecutive different button plays are commutative, which means to exchange the playing order will not change the output. This because if a light button is involved those two plays, then it's light would not be changed after two plays; if a light button only in involved one play action, then it's light would be always changed after two plays. Therefore any two button plays are commutative even they are not in consecutive order.
(2) Two consecutive same button plays will be equivalent to no play at all.

From those two notes, we conclude that for any solvable initial setting, each button only needs to be pressed at most once and the plays can be in any order.

§2. The linear equation system behind the game

We have two pictures below:

b_1	b_2	b_3	b_4	b_5
b_6	b_7	b_8	b_9	b_{10}
b_{11}	b_{12}	b_{13}	b_{14}	b_{15}
b_{16}	b_{17}	b_{18}	b_{19}	b_{20}
b_{21}	b_{22}	b_{23}	b_{24}	b_{25}

x_1	x_2	x_3	x_4	x_5
x_6	x_7	x_8	x_9	x_{10}
x_{11}	x_{12}	x_{13}	x_{14}	x_{15}
x_{16}	x_{17}	x_{18}	x_{19}	x_{20}
x_{21}	x_{22}	x_{23}	x_{24}	x_{25}

The numbers $b_1, b_2, \ldots, b_{25}$ represent the initial lights setting and the numbers x_1, $x_2, \ldots, x_{25}$ represent the button pressing plays. Each number b_i ($1 \le i \le 25$) is 1 or 0 such that 1 means light on and 0 means light off. Each number x_i ($1 \le i \le 25$) is also 1 or 0 such that 1 means press this button and 0 means no play action on this button. All numbers are in module group Z_2, satisfying:

addition rule $0 + 0 = 0$, $0 + 1 = 1$, $1 + 0 = 1$, $1 + 1 = 0$

subtraction rule $1 - 1 = 0$, $1 - 0 = 1$, $0 - 0 = 0$, $0 - 1 = 1$

When playing x_1, the light b_1 becomes $b_1 + x_1$. To make b_1 to be zero, we need plays x_2 and x_6 too. Therefore $b_1 + x_1 + x_2 + x_6 = 0$ and we have equation

$$x_1 + x_2 + x_6 = b_1$$

To make b_2 to be zero, we need plays x_1, x_2, x_3 and x_7. Therefore we have

$$x_1 + x_2 + x_3 + x_7 = b_2.$$

To make b_3 to be zero, we need to play x_2, x_3, x_4 and x_8. Therefore we have

$$x_2 + x_3 + x_4 + x_8 = b_3.$$

Similarly, we can get the following 22 equations:

$$x_3 + x_4 + x_5 + x_9 = b_4$$
$$x_4 + x_5 + x_{10} = b_5$$
$$x_1 + x_6 + x_7 + x_{11} = b_6$$
$$x_2 + x_6 + x_7 + x_8 + x_{12} = b_7$$
$$x_3 + x_7 + x_8 + x_9 + x_{13} = b_8$$
$$x_4 + x_8 + x_9 + x_{10} + x_{14} = b_9$$
$$x_5 + x_9 + x_{10} + x_{15} = b_{10}$$
$$x_6 + x_{11} + x_{12} + x_{16} = b_{11}$$
$$x_7 + x_{11} + x_{12} + x_{13} + x_{17} = b_{12}$$
$$x_8 + x_{12} + x_{13} + x_{14} + x_{18} = b_{13}$$

$$x_9 + x_{13} + x_{14} + x_{15} + x_{19} = b_{14}$$

$$x_{10} + x_{14} + x_{15} + x_{20} = b_{15}$$

$$x_{11} + x_{16} + x_{17} + x_{21} = b_{16}$$

$$x_{12} + x_{16} + x_{17} + x_{18} + x_{22} = b_{17}$$

$$x_{13} + x_{17} + x_{18} + x_{19} + x_{23} = b_{18}$$

$$x_{14} + x_{18} + x_{19} + x_{20} + x_{24} = b_{19}$$

$$x_{15} + x_{19} + x_{20} + x_{25} = b_{20}$$

$$x_{16} + x_{21} + x_{22} = b_{21}$$

$$x_{17} + x_{21} + x_{22} + x_{23} = b_{22}$$

$$x_{18} + x_{22} + x_{23} + x_{24} = b_{23}$$

$$x_{19} + x_{23} + x_{24} + x_{25} = b_{24}$$

$$x_{20} + x_{24} + x_{25} = b_{25}$$

Let ***A*** be the equation coefficient matrix, ***X*** be the column vector of plays $\{x_i\}$ and ***b*** be the column vector of lights initial setting $\{b_i\}$, then we have a linear equation system ***AX = b.***

The matrix *A* can be calculated by the following MATLAB commends:

```
>> A = zeros(25);
>> A(1,[1 2 6]) = 1;
>> A(2,[1:3 7]) = 1;
>> A(3,[2:4 8]) = 1;
>> A(4,[3:5 9]) = 1;
>> A(5,[4 5 10]) = 1;
>> A(6,[1 6 7 11]) = 1;
>> A(7,[2 6:8 12]) = 1;
>> A(8,[3 7:9 13]) = 1;
>> A(9,[4 8:10 14]) = 1;
>> A(10,[5 9 10 15]) = 1;
>> A(11,[6 11 12 16]) = 1;
>> A(12,[7 11:13 17]) = 1;
>> A(13,[8 12:14 18]) = 1;
>> A(14,[9 13:15 19]) = 1;
>> A(15,[10 14 15 20]) = 1;
>> A(16,[11 16 17 21]) = 1;
>> A(17,[12 16:18 22]) = 1;
>> A(18,[13 17:19 23]) = 1;
>> A(19,[14 18:20 24]) = 1;
>> A(20,[15 19:20 25]) = 1;
>> A(21,[16 21 22]) = 1;
>> A(22,[17 21:23]) = 1;
>> A(23,[18 22:24]) = 1;
>> A(24,[19 23:25]) = 1;
>> A(25,[20 24 25]) = 1;
```

$$A = \begin{bmatrix}
1&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&1&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&1&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&1&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&1&1&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&1&1&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&1&1&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&1&1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
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0&0&0&0&0&0&0&1&0&0&0&1&1&1&0&0&0&1&0&0&0&0&0&0&0\\
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0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&1&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&1&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&1&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&1&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1
\end{bmatrix}$$

The determinant of this matrix is 0, so matrix A is singular, which means the equation system $\boldsymbol{AX = b}$ might not have solution for some initial setting of $\{b_i\}$. Because each entry of A is either 1 or zero, calling Gauss-Jordan elimination operation on $\boldsymbol{A}$ would not create fractional numbers.

By call MATLAB command

>> G = mod(rref(A), 2);

We get matrix

$$G = \begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
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0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0\\
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0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix}$$

Therefore the rank(A) is 23 and the dimension of null space of A is 2.
Now consider the homogenous equation system $\boldsymbol{GX = \varnothing}$
Set $x_{24} = 1, x_{25} = 0$ or $x_{24} = 0, x_{25} = 1$, we obtain two independent vectors of the null space as the following:

$$v_1 = [0\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 0];$$
$$v_2 = [1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1];$$

Because A is symmetric, $\mathrm{null}(A) \perp \mathrm{range}(A)$. So the equation system $AX = b$ has solution if and only if $b \perp v_1$ and $b \perp v_2$ (in Z_2).

§3. Generate the solvable initial setting

The following MATLAB function will generate the solvable initial setting:

```
function b = generate()
    v1 = [0 1 1 1 0 1 0 1 0 1 1 1 0 1 1 1 0 1 0 1 0 1 1 1 0];
    v2 = [1 0 1 0 1 1 0 1 0 1 0 0 0 0 0 1 0 1 0 1 1 0 1 0 1];
    c = clock;
    rand('seed', c(6));
    while true
       b = mod(floor(rand(1, 25)*2), 2);
       if mod(dot(b,v1), 2)==1 | mod(dot(b,v2), 2)==1
           continue;
       end
       break
    end
end
```

Here we set seed for the random function **rand()** according to the clock such that we can get the different solvable initial setting at different time.

§ 3. Finding the solution for equation system $\boldsymbol{AX = b}$

Now assume that equation system $AX = b$ has solution. From linear algebra, the Gauss-Jordan elimination is equivalent to a multiplication by a matrix R from the left, which means that $RA=G$. Then we get $GX = Rb$. Because the last two rows of G are zeros, therefore the last two entries of Rb must be zeros too. So we can set $x_{24}=0$ and $x_{25}=0$, then $GX = I_{25}X = X$, we get a solution $X = Rb$. Other three solutions are $Rb+v_1$, $Rb+v_2$, $Rb+v_1+v_2$. Among them, the solution with the minimum sum (the regular sum) of entries will be the minimal playing solution.
The matrix R is the following

$$R = \begin{bmatrix}
0&1&1&1&0&0&0&1&0&1&0&0&0&1&1&0&0&0&0&1&0&0&0&0&0\\
1&1&0&1&1&0&1&0&0&0&0&0&1&1&1&0&0&0&1&0&0&0&0&0&0\\
1&0&1&1&1&1&0&1&1&0&0&0&1&1&0&1&1&1&1&1&0&1&0&0&0\\
1&1&1&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&1&0&0\\
0&1&1&0&1&1&0&0&0&0&1&0&1&0&1&0&0&1&0&1&1&1&0&0&0\\
0&0&1&0&1&0&1&1&0&1&0&0&1&0&0&0&0&0&1&1&0&0&0&0&0\\
0&1&0&1&0&1&1&0&1&1&0&0&0&1&0&1&1&1&0&0&0&1&0&0&0\\
1&0&1&0&0&1&0&1&1&0&0&0&0&0&1&1&0&1&0&1&1&0&1&0&0\\
0&0&1&0&0&0&1&1&1&0&1&0&0&1&1&1&0&0&1&0&0&1&1&0&0\\
1&0&0&0&0&1&1&0&0&0&1&0&1&0&1&0&1&1&0&1&0&0&1&0&0\\
0&0&0&0&1&0&0&0&1&1&0&0&1&0&1&1&1&1&1&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&1&0&0\\
0&1&1&0&1&1&0&0&0&1&1&0&1&1&0&0&0&1&0&0&1&1&0&0&0\\
1&1&1&0&0&0&1&0&1&0&0&0&1&1&1&0&0&0&1&0&0&0&0&0&0\\
1&1&0&0&1&0&0&1&1&1&1&0&0&1&1&1&1&0&1&0&1&0&0&0&0\\
0&0&1&0&0&0&1&1&1&0&1&0&0&0&1&1&0&1&0&1&1&0&1&0&0\\
0&0&1&1&0&0&1&0&0&1&1&1&0&0&1&0&1&1&1&0&0&0&1&0&0\\
0&0&1&0&1&0&1&1&0&1&1&0&1&0&0&1&1&0&1&1&1&0&0&0&0\\
0&1&1&0&0&1&0&0&1&0&1&0&0&1&1&0&1&1&1&0&0&0&1&0&0\\
1&0&1&0&1&1&0&1&0&1&0&0&0&0&0&1&0&1&0&1&1&0&1&0&0\\
0&0&0&1&1&0&0&1&0&0&0&1&1&0&1&1&0&1&0&1&1&1&0&0&0\\
0&0&1&1&1&0&1&0&1&0&1&1&1&0&0&0&0&0&0&0&1&1&1&0&0\\
0&0&0&1&0&0&0&1&1&1&0&1&0&0&0&1&1&0&1&1&0&1&0&0&0\\
0&1&1&1&0&1&0&1&0&1&1&1&0&1&1&1&0&1&0&1&0&1&1&1&0\\
1&0&1&0&1&1&0&1&0&1&0&0&0&0&0&1&0&1&0&1&1&0&1&0&1
\end{bmatrix}$$

To matrix *R* can be generated by the following MATLAB function:

```
function R = GJ(A)
    [m n] = size(A);
    A = round([A, eye(m)]);
        % add identity
    for i=1:m
        if A(i, i)==0
            for k = i+1:m
                if A(k,i)~=0
                   A([i k],:) = A([k i],:);          % swap two rows
                   break;
                end
            end
        end
        if A(i, i)==0
            break
                                  % stop loop
        end
        for k=[1:i-1 i+1:m]
           A(k, :) = mod(A(k, :)+ A(i, :)*A(k, i),2);
        end
    end
    R = A(:,n+1:n+m);
               % read answer
end
```

To get minimal play solution, call command

>> y = R*b;
>> y1 = mod(y + v1, 2);
>> y2 = mod(y + v2, 2);
>> y2 = mod(y + v1 + v2, 2);

```
>> if sum(y) < sum(y1)
                y = y1;
      end
>> if sum(y) < sum(y2)
                y = y2;
      end
>> if sum(y) < sum(y3)
                y = y3;
      end
```


§4. Window application for the game

Based on the above linear algebraic analyzing, we have designed a window application for the ***Lights Out*** game in Microsoft DOT NET C# language. Users can setup the initial lights manually or choose auto setting. The users also can choose either hand play or auto play. The following are some screen shots

The complete code and the executable application can be downloaded from web site

http://sunoas.com/lightsout.

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References

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