Some Monoids of Integer Partitions

D. Singh, Ph.D.†
J. N. Singh, Ph.D. ‡

Abstract:

In this short communication, certain monoids of integer partitions are developed. In particular, a couple of new operations are introduced which would be applicable even for constructing some monoids of set partitions as well as for partitions of hybrid sets like multiset, fuzzy set, soft set, etc.

Key Words: Integer, Partitions, Monoids

1. Introduction

The mathematics of integer partitions has a long and tortuous history (see [4], for a recent reference). In view of the fact that certain monoids, consisting of partitions of a set, have useful applications in computer arithmetic, formal languages and sequential machines (see [2,5], for example), this paper intends to define a couple of new operations on the class of integer partitions which would give rise to certain monoids.

2. Monoids of Integer Partitions

Let us represent a positive integer n as an n-set viz., $n = \{ 0,\ 1, 2, \cdots,\ n-1 \}$. Let $\overline{P(n)}$ denote the set of all partitions of n, and $P = \{ P_1, P_2, \cdots \}$; $Q = \{ Q_1, Q_2, \cdots \}$ be two partitions of n.

Let us define a binary operation $*$ on $\overline{P(n)}$ such that $P * Q$ consists of all nonempty intersections of every element of P with every element of Q. It is immediate to see that the operation $*$ is both associative and commutative, and the partition $\overline{\{ 0,\ 1,\ 2, \cdots, n-1 \}}$, consisting of a single block, is the identity of the operation $*$. Thus, $\left\langle \overline{P(n)}, * \right\rangle$ or $\left\langle \overline{P(n)}, *, \overline{\{ 0,\ 1, 2, \cdots,\ n-1 \}} \right\rangle$ is a monoid. It may also be observed that every element $T \in \overline{P(n)}$ is idempotent with respect to the operation $*$, since $T * T = T$ holds. The operation $*$ on $\overline{P(n)}$ is called the *product of partitions*.

Alternatively, $P * Q$ can be thought of as a partition corresponding to the equivalence relation $R_1 \cap R_2$, where R_1 and R_2 are the equivalence relations corresponding to P and Q, respectively. Thus, any two parts λ and λ' occur in the same block of $P*Q$ if they occur in the same block of P and, also in the same block of Q.

It is interesting to see that another binary operation on $\overline{P(n)}$ can be defined with respect to which it is a monoid. Let us denote it by $\oplus$ and define as follows:

For any $P, Q \in \overline{P(n)}$, a subset T of n belongs to $P \oplus Q$ if

i. T is the union of one or more elements of P;

ii. T is the union of one or more elements of Q; and

iii. no subset of T satisfies (i) and (ii) except T itself.

It is straightforward to see that $\left\langle \overline{P(n)}, \oplus \right\rangle$ is a monoid. The operation $\oplus$ is called the *sum of partitions.*

Alternatively, $P \oplus Q$ can be thought of as a partition corresponding to the equivalence relation $R_1 \cap R_2$, where R_1 and R_2 are the equivalence relations corresponding to P and Q respectively. Thus, any two parts λ and λ' occur in the same block of $P \oplus Q$ if there exists a sequence of parts $\lambda, \lambda_1, \lambda_2, \cdots, \lambda_k, \lambda'$ such that each pair of successive parts in the sequence is in the same block of P or Q.

Example

Let $n = 5 = \{0, 1, 2, 3, 4\}$, then

$\overline{P(5)} = \{\{\overline{0, 1, 2, 3, 4}\}, \{\overline{0, 1, 2, 3}, \overline{4}\}, \{\overline{0, 1, 2}, \overline{3, 4}\}, \{\overline{0, 1, 2}, \overline{3}, \overline{4}\}, \{\overline{0, 1}, \overline{2, 3}, \overline{4}\}, \{\overline{0, 1}, \overline{2}, \overline{3}, \overline{4}\}, \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\}\}$
corresponding to
$\{\{\overline{5}\}, \{\overline{4}, \overline{1}\}, \{\overline{3}, \overline{2}\}, \{\overline{3}, \overline{1}, \overline{1}\}, \{\overline{2}, \overline{2}, \overline{1}\}, \{\overline{2}, \overline{1}, \overline{1}, \overline{1}, \}, \{\overline{1}, \overline{1}, \overline{1}, \overline{1}, \overline{1}\}\}$.

Let $P = \{\overline{0, 1, 2}, \overline{3}, \overline{4}\}$ and $Q = \{\overline{0, 1}, \overline{2, 3, 4}\}$ corresponding to $\{\overline{3}, \overline{1}, \overline{1}\}$ and $\{\overline{2}, \overline{2}, \overline{1}\}$, then

$P * Q = \{\overline{0, 1}, \overline{2}, \overline{3}, \overline{4}\}$, which corresponds to $\{\overline{2}, \overline{1}, \overline{1}, \overline{1}, \} \in \overline{P(5)}$.

Also, $P \oplus Q = \{\overline{0, 1, 2, 3}, \overline{4}\}$ which corresponds to $\{\overline{4}, \overline{1}\} \in \overline{P(5)}$.

Note that, unlike set partitions, the number of partitions of an n-set is strictly larger than the number of partitions of an integer n, for n>2. It is because the set of integer partitions of n are ordered (under antilexicographic ordering), where as the set partitions of an n-set are unordered, in general (see [1, 3, 4], for various details).

Concluding Remarks

It seems promising that some other algebraic structures could be described by way of discovering new operations suitably defined on the class of integer partitions.

† *D. Singh, Ph.D.* Mathematics Department, A. B. U. Zaria, Nigeria
‡ *J. N. Singh, Ph.D.* Department of Mathematics & Computer Science
Barry University, FL, USA

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