An Episode of the Story of the Cubic Equation: The del Ferro-Tartaglia-Cardano's Formulas

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Abstract

In this paper, I discuss the contributions of del Ferro, Tartaglia, and Cardano in the development of algebra, specifically determining the formula to solve cubic equations in one variable. To contextualize their contributions, brief biographical sketches are included. A brief discussion of the influence of *Ars Magna*, Cardano's masterpiece, in the development of algebra is also included.

Introduction

Our mathematical story takes place in the Renaissance era, specifically in sixteenth-century Italy. The Renaissance was a period of cultural, scientific, technology, and intellectual accomplishments. Vesalius's *On the Structure of the Human Body* revolutionized the field of human anatomy while *Copernicus' On the Revolutions of the Heavenly Spheres* transformed drastically astronomy; both texts were published in 1543. Soon after the publication of these two influential scientific treatises, a breakthrough in mathematics was unveiled in a book on algebra whose author, Girolamo Cardano (1501-1576), exemplifies the common notion of Renaissance man. As a universal scholar, Cardano was a physician, mathematician, natural philosopher, astrologer, and interpreter of dreams.

Cardano's book, *Ars Magna* (*The Great Art*), was published in 1545. As soon as *Ars Magna* appeared in print, the algebraist Niccolo Fontana (1499-1557), better known as Tartaglia – the stammerer– was outraged and furiously accused the author of deceit, treachery, and violation of an oath sworn on the Sacred Gospels. The attacks and counterattacks set the stage for a fierce battle between the two mathematicians. What does *Ars Magna* contain that triggered one of the greatest feuds in the history of mathematics? (Helman, 2006). To understand the vehement and displeasing dispute, one must recount the state of mathematical knowledge at the start of the sixteenth century and the background of the main characters. We begin with the background of the problem.

Algebra of the Beginning of the Sixteenth Century

In 1494, the Italian mathematician Luca Pacioli (ca. 1445-1509) published in Venice the remarkable compendium *Summa de Arithmetica, Geometria, Proportioni, e Proportionalita*. In it, the author discussed the basic principles of algebra with special emphasis on solving linear and quadratic equations. The *Summa* also contains a discussion of solutions of higher-degree equations. In particular, Pacioli noticed that "it has not been possible until now to form general rules" (p. 47) to solve cubic and quartic equations. In addition, Pacioli mentions the impossibility of solving quadratic equations next to the unsolved classical problem of squaring the circle. Some members of the Italian mathematical community took up the challenge to find a general procedure to determine the solution of a cubic equation, that is, an equation of the form $ax^3 + bx^2 + cx + d = 0$.

During this time, the mathematicians had not accepted yet the use of 0 as a number and neither the use of negative numbers. Hence, setting an equation equal to zero was unthinkable and only the positive solutions of equations were considered. While today we have only one form of the general cubic equation ($ax^3 + bx^2 + cx + d = 0$ or $x^3 + nx^2 + px = q$), 16th century mathematicians investigated 13 different cases of cubic equations separately: 7 types of cubic equations with all the terms ($x^3 + nx^2 + px = q$, $x^3 + nx^2 + q = px$, $x^3 + px + q = nx^2$, $x^3 + nx^2 = px + q$, $q\,x^3 + px = nx^2 + q$, $x^3 + q = nx^2 + px$, $x^3 = nx^2 + px + q$), 3 cases of the cubic equation lacking the linear term ($x^3 + nx^2 = q$, $x^3 + q = nx^2$, $x^3 = nx^2 + q$), and 3 special cubic equations without the quadratic term ($x^3 + px = q$, $x^3 + q = px$, $x^3 = px + q$). Of course, the three cases without the independent term ($x^3 + nx^2 = px$, $x^3 + px = nx^2$, $x^3 = nx^2 + px$) could be reduced to a quadratic equation.

In 1509, Pacioli published his translation of Euclid's *Elements*, which written in Italian, became a source of mathematical knowledge for his countrymen not versed in Latin. Italian scholars, continuing Euclid's work as discussed in his tenth book, extensively studied sum and differences of square roots such as $a + \sqrt{b}$, $\sqrt{a} + \sqrt{b}$, $\sqrt{a} - \sqrt{b}$. The sums were called binomials (binomiums in Latin) and the differences remainders (apotemes in Latin). They determined the solution to the quadratic equation $x^2 + px = q$ using the formula $\sqrt{\left(\frac{p}{2}\right)^2 + q} - \frac{p}{2}$. This suggests that a quadratic equation has a solution of the form $\sqrt{a} - b$ or $\sqrt{a} + b$ (for $x^2 = px + q$). Also, scholars of that time may have noticed that the difference of squares roots represented as $x = \sqrt{a + \sqrt{b}} - \sqrt{a - \sqrt{b}}$ when squared results in the quadratic equation $x^2 = 2a - 2\sqrt{a^2 - b}$, which lacks the first power of x.

Del Ferro

Our story begins with Scipione del Ferro (1465-1526), a mathematician who lectured in arithmetic and geometry at the University of Bologna. Around 1515, the talented del Ferro made the first critical breakthrough by discovering a formula for solving a case of the so-called "depressed cubic," a specific cubic equation that lacks its quadratic term. In other words, a cubic equation of the form $ax^3 + cx + d = 0$ or of the forms $x^3 + px = q$, $x^3 + q = px$, or $x^3 = px + q$, where p and q are positive numbers. There is no general agreement among scholars whether del Ferro solved all the cases of the depressed cubic or only the case $x^3 + px = q$. In any event, this was a significant advance in the search for a formula to solve the general cubic equation. Del Ferro's algorithm to solve the equation $x^3 + px = q$ translates into modern notation as follows

$$x = \sqrt[3]{\sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3} + \frac{q}{2}} - \sqrt[3]{\sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3} - \frac{q}{2}}$$

However, del Ferro did not provide any hint of how he derived the formula. One approach that del Ferro may have used, according to Calinger (1999), is to consider the sum or difference of cubic roots in the same way that scholars at that time may have considered the sum and differences of square roots to produce quadratic equations, as mentioned above. The process may have been as follows

Cubing the difference given by $x = \sqrt[3]{a+\sqrt{b}} - \sqrt[3]{a-\sqrt{b}}$ results in

$$x^3 = (a+\sqrt{b}) - 3(a+\sqrt{b})^{\frac{2}{3}}\left(a-\sqrt{b}\right)^{\frac{1}{3}} + 3(a+\sqrt{b})^{\frac{1}{3}}\left(a-\sqrt{b}\right)^{\frac{2}{3}} - \left(a-\sqrt{b}\right)$$

After simplifying and factoring, this equation reduces to

$$x^3 = 2\sqrt{b} - 3(a+\sqrt{b})^{\frac{1}{3}}\left(a-\sqrt{b}\right)^{\frac{1}{3}}\left[\left(a+\sqrt{b}\right)^{\frac{1}{3}} - \left(a-\sqrt{b}\right)^{\frac{1}{3}}\right]$$

And then to

$$x^3 = 2\sqrt{b} - 3(a+\sqrt{b})^{\frac{1}{3}}\left(a-\sqrt{b}\right)^{\frac{1}{3}}x$$

Rearranging terms becomes

$$x^3 + 3(a+\sqrt{b})^{\frac{1}{3}}\left(a-\sqrt{b}\right)^{\frac{1}{3}}x = 2\sqrt{b}$$

As we notice, this last equation is of the form $x^3 + px = q$ with

$$p = 3(a+\sqrt{b})^{\frac{1}{3}}\left(a-\sqrt{b}\right)^{\frac{1}{3}} = 3\left(a^2-b\right)^{\frac{1}{3}} \text{ and } q = 2\sqrt{b}$$

Solving for a and b in terms of p and q one gets

$$b = \left(\frac{q}{2}\right)^2 \text{ and } a = \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}$$

Substituting these expressions into the first equation yields the cubic formula

$$x = \sqrt[3]{\sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3} + \frac{q}{2}} - \sqrt[3]{\sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3} - \frac{q}{2}}$$

Working in an academic environment where “publish or perish” seems to be a continuous pressure to sustain or further one’s career, current scholars may be surprised to learn that del Ferro, as other scholars of his time, did not publish his discovery. During the Renaissance, academic appointments were mostly temporary, subject to renewal, and held by scholars who had to prove their talent and reputation by winning public contests against other scholars. Being the only one knowing how to solve depressed cubic equations, del Ferro would prevail in a public competition by challenging the opponent to solve problems involving these types of equations. Even if del Ferro were unable to solve some of his opponent’s problems, he could feel confident that his adversary would be stumped by his problems.

Del Ferro recorded his solution to the depressed cubic equation and some of his other contributions to mathematics in a notebook. When he died in 1526, del Ferro passed this notebook on to his son-in-law, Aniballe della Nave, and to Antonio Fior, one of his (rather poor) students. Fior felt that he had a new powerful weapon in his possession and, wanting to secure a position as a teacher of mathematics, he issued a challenge to Nicolo Fontana, a noted scholar who bragged in 1535 that he could solve cubic equations lacking the linear term, that is, equations of the form $x^3 + n\,x^2 = q$.

Tartaglia

Niccolo Fontana, best known as Tartaglia, was born in Brescia, in northern Italy, in 1499 or 1500. In 1512 his home town was sacked by the French. Amidst the slaughter, Niccolo was almost killed by a French soldier who cut his jaw and palate leaving the teenager boy disfigured and unable to speak clearly. He was nicknamed Tartaglia- the Stammerer - and it is by this epithet that he is best known today.

Tartaglia was for the most part an autodidact who taught himself mathematics and mechanics. He left Brescia to settle in Verona and then in Venice to earn his living as a mathematics teacher. As other teachers and scholars, he had to participate in public contests and debates to keep his name and reputation before the public. Due to his success in such debates, his reputation gradually grew and he became a distinguished mathematician. Thus, challenging and defeating him in a public contest would bring fame and reputation to the winner. Tartaglia became the perfect target for Fior who was remarkably confident that his knowledge of how to solve depressed cubic equations would be enough to guarantee the downfall of Tartaglia.

When the writing mathematical contest was set up, each scholar submitted 30 problems for the other to solve. The winner would be the one who solved the most problems after a certain period of time (30 days according to some sources, 40 or 50 according to others). The loser would pay for a banquet for 30 people.

Tartaglia provided Fior with a list of problems on different mathematical topics. In contrast, every one of Fior's problems was reduced to solving a depressed cubic of the type he could solve, placing Tartaglia in a bind. One of the problems that Fior proposed to Tartaglia is the following: A man sells a sapphire for 500 ducats, making a profit of the cubic root of his capital. How much is this profit? (Fauvel & Gray, 1987, p. 254)

Facing this problematic situation, Tartaglia began a frantic search to find the general rule to solve depressed cubic equations. His strenuous efforts paid off when, in the early hours of February 13, seven days before the deadline, inspiration came over him and Tartaglia managed to discover the method to solve these types of equations (unknowns and cubes equal to number). Tartaglia solved Fior's problems in less than two hours, whereas his weak opponent was not able to solve Tartaglia's problems because, as it turned out, Fior's mathematical knowledge did not extend much beyond solving these kinds of equations. Tartaglia's victory was unquestionable, but, magnanimously, he did not claim the banquet.

Again, we can only guess how Tartaglia arrived at his solution, which is equivalent to

$$\sqrt[3]{\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}+\frac{q}{2}}-\sqrt[3]{\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}-\frac{q}{2}}$$

One way could have been using del Ferro's approach, as argued by Katscher (2011). Another method, suggested by Katz and Parshall (2014), could have been comparing $x^3 + px = q$ with the identity $(u-v)^3 + 3uv(u-v) = u^3 - v^3$.

Setting $x = u - v$, $p = 3uv$, and $q = u^3 - v^3$ amounts to finding u and v in terms of p and q. Using the following algebraic identity, as suggested by Katz and Parshall (2014),

$$\left(\frac{u^3+v^3}{2}\right)^2 = \left(\frac{u^3-v^3}{2}\right)^2 + u^3v^3$$

del Ferro could have reasoned as follows. Making appropriate substitutions, one obtains

$$\left(\frac{u^3+v^3}{2}\right)^2 = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3$$

Taking the square root of both sides of this equation, it follows that

$$\frac{u^3+v^3}{2} = \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}$$

Using the fact that $\frac{u^3-v^3}{2} = \frac{q}{2}$ results in

$$u^3 = \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3} + \frac{q}{2}$$

Taking the cubic root of both sides of this equation results in

$$u = \sqrt[3]{\sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3} + \frac{q}{2}}$$

Now, using the fact that $u^3 - v^3 = q$ yields

$$v^3 = u^3 - q$$

or

$$v^3 = \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3} + \frac{q}{2} - q$$

Simplifying and taking the cubic root produces

$$v = \sqrt[3]{\sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3} - \frac{q}{2}}$$

and consequently,

$$x = u - v = \sqrt[3]{\sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3} + \frac{q}{2}} - \sqrt[3]{\sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3} - \frac{q}{2}}$$

News of Tartagli's triumph eventually reached Girolamo Cardano, one of the most interesting, extraordinary, and bizarre personages in the entire history of mathematics.

Cardano

In his autobiography *De Vita Propria Liber* (*The Book of My Life*), Cardano (1576/2002) recounts that, "although various abortive medicine … were tried in vain" he was born on September 24, 1501 after having been "literally torn from [his] mother's womb" because she "had been in labor for three entire days" and yet he survived. If what he heard is right, Cardano's birth was unwelcome because he was the illegitimate child of Fazio Cardano and Chiara Micheria. Even with this shaky start, Cardano would become one of the most respected doctors, mathematicians, philosophers, and astrologers of his time. In 1552 for example, he traveled to Scotland to treat the Archbishop of St Andrews, John Hamilton, who had been suffering from asthma for ten years and the condition was worsening. The archbishop recovered completely and this success solidified Cardano's medical reputation.

Having heard that Tartaglia had a method for solving some cubic equations, Cardano, who at this time was a public lecturer at the Piatti Foundation in Milan and was writing a book on algebra, was intrigued because he took Pacioli's words that the solution of the cubic was impossible. For several years Cardano tried to discover the elusive formula, but was unsuccessful. In 1539, he contacted Tartaglia through an intermediary who told Tartaglia that the formulas could be included in a book that Cardano was planning to publish soon. Tartaglia declined. Again and again Cardano wrote to him supplicating him to reveal the secret, and over and over Tartaglia refused. After much insistence, on the promise that Cardano would introduce him to the governor of Milan, one of Cardano's patrons, Tartaglia accepted an invitation to Milan. On March 25, 1539, at Cardano's house, Tartaglia revealed the secret of solving cubic equations, after making Cardano swear that he was not going to divulge it. Tartaglia's report of the oath sworn by Cardano was:

I swear to you, by God's holy Gospels, and as a true man of honour, not only never to publish your discoveries, if you teach me them, but I also promise you, and I pledge my faith as a true Christian, to note them down in code, so that after my death no one will be able to understand them. If you want to believe me now, then believe me, if not, leave it be. (Fauvel & Gray, 1987, pp. 255).

Tartaglia encrypted the solution in a poem whose first verse reads as follows:

When the cube and its things near ($x^3 + px$)
Add to a new number, discrete, ($x^3 + px = q$)
Determine two new numbers different
By that one; this feat ($u - v = q$)
Will be kept as a rule
Their product always equal, the same, (uv)
To the cube of a third
Of the number of things named. ($uv = \left(\frac{p}{3}\right)^3$)
Then, generally speaking,
The remaining amount
Of the cube roots subtracted ($x = \sqrt[3]{u} - \sqrt[3]{v}$)
Will be your desired count (Katz, 2009, p. 400)

Let's solve the problem about the cost of the sapphire stated above using Tartaglia's process. First, the solution can be represented as $x^3 + x = 500$. Following Tartaglia's method, we need to find two numbers u and v such that $u - v = 500$ and $uv = \left(\frac{1}{3}\right)^3 = \frac{1}{27}$. Solving this system of equations is equivalent to solving the quadratic equation $v^2 + 500v = \frac{1}{27}$, whose positive solution is found by applying the quadratic formula $v = \sqrt{\left(\frac{p}{2}\right)^2 + q} - \frac{p}{2} = \sqrt{\left(\frac{500}{2}\right)^2 + \frac{1}{27}} - 250 \approx 0.0000740741$. The solution is then now given by $x = \sqrt[3]{u} - \sqrt[3]{v} \approx \sqrt[3]{500.0000740741} - \sqrt[3]{0.0000740741} \approx 7.895$.

The complete poem can be found in Fauvel and Gray (1987) or Katscher (2011). In the remaining verses of the poem, Tartaglia tells Cardano how to solve the cubics $x^3 = px + q$ and $x^3 + q = px$. Translating the poem into the formula for $x^3 = px + q$ results in

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}}$$

In this case, however, evaluating the expression under the square root may sometimes not be possible with Cardano's mathematics because $(q/2)^2 - (p/3)^3 < 0$ amounts to using square roots of negative numbers to find a solution of the equation $x^3 = px + q$. This could be fine if it only happened when the cubic equations did not have real roots. Cardano, before he wrote his *Ars Magna*, applied Tartaglia' method to the equation $x^3 = 15x + 4$ yielding

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

From the appearance of this sophisticated-looking expression, we may conclude that the equation $x^3 = 15x + 4$ has no real solution. Yet, as we can easily verify, this equation has three real roots: $x = 4$, $x = -2 + \sqrt{3}$, and $x = -2 - \sqrt{3}$.

Cardano was at loss and asked Tartaglia about it:

I have sent to enquire after the solution to various problems for which you have given me no answer, one of which concerns the cube equal to an unknown plus a number. I have certainly grasped this rule, but when the cube of one-third of the coefficient of the unknown is greater in value than the square of one-half of the number, then, it appears, I cannot make it fit into the equation. (O'Connor & Robertson).

As we can notice from this paragraph, Cardano is also stating the conditions under which the formula would involve the use of square roots of negative numbers, that is, complex numbers. Tartaglia, of course, did not understand complex numbers either so he simply replied to Cardano that "you have not mastered the true way of solving problems of this kind, and indeed I would say that your methods are totally false" (O'Connor & Robertson).

Cardano expanded Tartaglia's ideas and was able to solve all the cases of the cubic, including the general third-degree equation of the form $x^3 + nx^2 + px =$

q. At this point in the story, another character enters into the drama: Ludovico Ferrari.

Ferrari

Ludovico Ferrari was born in 1522 in Bologna. On November 30, 1536, when Ludovico was just 14 years old, Cardano took him as a servant. Cardano almost immediately realized that Ludovico was exceptionally gifted and decided to teach him mathematics. Soon Cardano also shared with him Tartaglia's secret method to solve depressed cubics. Working together, they made remarkable progress. Cardano solved the general cubic equation by means of a transformation that reduces the general cubic equation to a depressed cubic. In 1540, Ferrari, applying a similar substitution, reduced the quartic equation to a depressed quartic equation, which he then manipulated to transform both sides of it into perfect squares. This transformation involved a cubic equation that could be solved now using Cardano and Tartaglia's method. In addition, Cardano realized the necessity of imaginary numbers to deal with the irreducible case of the cubic (i.e., $x^3 = px + q$). In fact, Cardano constructs examples with the sole goal of showing that this irreducible case of the cubic involves imaginary numbers that he manipulates as if they followed all the properties of common numbers. Cardano and Ferrari were now in possession of powerful mathematical discoveries that had eluded mathematicians for centuries. The two scholars, however, could not publish their discoveries without breaking Cardano's solemn promise.

Cardano kept his promise for several years but he must have slowly realized that Fior could have not issued the challenge without knowing how to solve the equation and, given that Fior was not a talented mathematician, it meant that del Ferro had discovered it and then passed it on to his student. In 1543, Cadano and Ferrari traveled to Bologna where they looked through del Ferro's papers. Their inspection led them to conclude that del Ferro had discovered the solution of one of the forms of the cubic equation. To Cardano, the implications of the discovery were clear: he was no longer bound by the oath. And so in 1545, Cardano published his great masterpiece, *Ars Magna*, which consisted of 40 chapters. With most scrupulous care, he clearly acknowledged that the solution of the equation $x^3 + px = q$ had been discovered by del Ferro and rediscovered independently by Tartaglia, and that he himself had extended the solution to equations $x^3 = px + q$ and $x^3 + q = px$. He also credits Ferrari with having solved the quartic equation.

Ars Magna

Once *Ars Magna* appeared in print, it reached scholars all over Europe, including Tartaglia. Tartaglia was in rage. In his eyes, Cardano had violated a sacred oath in spite of the fact that he (Cardano) had not taken credit as the original discoverer. The following year Tartaglia published a book entitled "*New Problems and Inventions*" in which he recounts his side of the story. Ferrari, not Cardano, replied to the accusations challenging Tartaglia to a public competition. Tartaglia initially refused because Ferrari was still a relatively unknown mathematician again whom even a triumph would do little practical good. The two men exchanged letters full of charges and insults for about a year. Suddenly, in 1548, Tartaglia received an offer of a professorship on the condition that he wins a contest against Ferrari. Expecting to defeat Ferrari, Tartaglia accepted. The debate took place in Milan on August 10, 1548. Ferrari understood the solutions of cubics and quartics while Tartaglia had not mastered these chapters of the *Ars Magna*. Thus, the contest culminated with Tartaglia's defeat as he withdrew from the competition.

Ars Magna contains a complete description of how to solve any type of cubic equation and provides geometric arguments to justify why the method works. Cardano's theory of the solution of the cubic starts in chapter XI: On the cube and First Power Equals to the Number. Following the tradition, Cardano did not give a general proof of the solution to this type of equation but rather uses a specific example $x^3 + 6x = 20$ to justify the general rule. Cardano's solution reads as follows:

Cube one-third the coefficient of x; add to it the square of one-half the constant of the equation; and take the square root of the whole. You will duplicate this, and to one of the two you add one-half the number you have already squared and from the other you subtract one-half the same. You will then have a binomium and its apoteme. Then, subtracting the cube root of the apoteme from the cube root of the binomium, the remainder or that which is left is the value of x (pp. 98-99).

According to Cardano's recipe to solve $x^3 + 6x = 20$, the first step is to cube a third of the coefficient of x to obtain $\left(\frac{6}{3}\right)^3 = 8$. Second, take the square of one-half 20 (100) and add it to 8 to yield 108. Third, take the square root of 108. Fourth, add and subtract half of the constant term (10) to get $\sqrt{108} + 10$ and $\sqrt{108} - 10$. Finally, the solution is the difference of the cube root of these two last numbers, namely, $x = \sqrt[3]{\sqrt{108}+10} - \sqrt[3]{\sqrt{108}-10}$. Cardano noticed that $\sqrt[3]{\sqrt{108}+10} - \sqrt[3]{\sqrt{108}-10} = 2$ stating that "is perfectly clear if it is tried out." (p. 100).

To justify his procedure, Cardano imagined a cube (Branson, 2013; Duhnam, 1990; Laubenbacher & Pengelly, 1999) with edge u partitioned into six parallelepipeds, as shown in figure 1. Here u and v are auxiliary variables to be determined. The volume of the cube with edge u can be expressed as the sum of the six parallelepipeds,

$$u^3 = v^3 + v^2\left(u-v\right) + 2uv(u-v) + \left(u-v\right)^3 + v\left(u-v\right)^2$$

Rearranging the terms and factoring $(u - v)$ produces

$$u^3 - v^3 = \left(u-v\right)^3 + (u-v)(v^2 + 2uv + uv - v^2)$$

$= (u - v)^3 + 3uv\,(u - v)$

Of course, this identity can nowadays be derived in a straightforward way using simple algebra. Notice however, that following Euclid's tradition this approach was not available to Cardano. Comparing the identity $(u - v)^3 + 3uv\,(u - v) = u^3 - v^3$ with the equation $x^3 + px = q$ led Cardano to conclude that letting $x = u - v$, $p = 3uv$, and $q = u^3 - v^3$ would transform the problem of solving for x into the equivalent problem of determining u and v (in terms of p and q). Once u and v are determined, the value of x can be determined as well. *Ars Magna* does not discuss a derivation of representing u and v in terms of p and q. Rather, its author simply provided the verbal rule with a specific example, as presented above. The derivation is as follows: From $3uv = p$, we obtain $u = \frac{p}{3v}$, which we then substitute in the second expression to yield

$$\frac{p^3}{27v^3} - v^3 = q$$

Multiplying both sides of this equation by v^3 and rearranging the terms produces

$$v^6 + q v^3 - \frac{p^3}{27} = 0$$

This equation is a quadratic equation in v^3:

$$\left(v^3\right)^2 + q\left(v^3\right) = \frac{p^3}{27}$$

Applying the quadratic formula to the last equation yields

$$v^3 = \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3} - \frac{q}{2}$$

Now, taking the cubic root produces

$$v = \sqrt[3]{\sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3} - \frac{q}{2}}$$

Now, using the fact that $u^3 - v^3 = q$ yields

$$u^3 = q + v^3$$

$$= \frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

$$u = \sqrt[3]{\sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3} + \frac{q}{2}}$$

We now have Cardano's implicit formula to solve equations of the type $x^3 + px = q$:

$$x = u - v = \sqrt[3]{\sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3} + \frac{q}{2}} - \sqrt[3]{\sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3} - \frac{q}{2}}$$

Cardano was able to solve the three types of depressed cubics. What about the general cubic equation of the form $ax^3 + bx^2 + cx + d = 0$ or $x^3 + n\,x^2 + px = q$? It was Cardano's great discovery that a general cubic equation could be transformed into a depressed cubic by a suitable substitution, namely $x = y - (b/3a)$ (Burton, 2011; Dunham, 1990). The process is as follows:

Substituting $x = y - b/3a = y - n/3$ in $x^3 + n\,x^2 + px = q$ yields

$$\left(y - \frac{n}{3}\right)^3 + n\left(y - \frac{n}{3}\right)^2 + p\left(y - \frac{n}{3}\right) = q$$

Expanding the left side of this equation becomes

$$y^3 - 3y^2\frac{n}{3} + 3y\frac{n^2}{9} - \frac{n^3}{27} + ny^2 - 2y\frac{n^2}{3} + \frac{n^3}{9} + py - \frac{pn}{3} = q$$

or

$$y^3 - ny^2 + \frac{n^2}{3}y - \frac{n^3}{27} + ny^2 - \frac{2n^2}{3}y + \frac{n^3}{9} + py - \frac{pn}{3} = q$$

Simplifying, this becomes

$$y^3 + \left(\frac{3p - n^2}{3}\right)y = \frac{9pn - 2n^3 + 27q}{27}$$

This equation can be solved using the formula for depressed cubics with $n = \frac{3p - n^2}{3}$ and $q = \frac{9pn - 2n^3 + 27q}{27}$.

Discussion and Conclusion

The formula to solve third-degree equations in one variable are commonly referred to as Cardano's formula or technique (Berlinghoff & Gouvêa, 2004; Burton, 2011; Calinger, 1999; Inving, 2013; Krantz, 2010; Laubenbacher & Pengelley, 1999). But as we can see, the formula was actually first discovered by del Ferro and then independently by Tartaglia. Cardano developed and justified the corresponding formula for every possible case of a cubic equation. Nowadays, of course, we have only one cubic formula. To do justice to all mathematicians who contributed to its development, the formula should be called del Ferro-Tartaglia-Cardano's formula.

Ars Magna played a significant role in the development of algebra, including modern algebra. As Felix Klein wrote “This work, which is of great value, contains the germ of modern algebra, surpassing the bounds of ancient mathematics” (Cited in Gindikin, 1988). To start, *Ars Magna*, took the art of solving equations to new heights. It contains significant results that neither ancient nor Easter mathematicians knew. Specifically, it contains the complete solution of solving *general* cubic and quartic equations in one variable by radicals, along with geometric arguments to justify the appropriate method. Cardano went beyond del Ferro and Tartaglia’s initial contributions.

Second, *Ars Magna* includes some examples of when the formula involves the square roots of negative numbers (in modern notation when the discriminant is negative). It also discusses the paradoxical case of a cubic having three real solutions and yet the formula involves the square root of negative numbers. Cardano clearly realized the existence of what we now call complex numbers, even though he ended up dismiss them, except in one problem. Cardano realized that if one manipulates the expressions involved in said problem, *putting aside the mental tortures involved* (Cardano, 1945/1968), one can obtain its solution, a situation that he called *truly sophisticated*. Thus, is the first mathematician who introduced complex numbers into algebra (van der Waweden, 1983; Varadarajan, 1998). By bringing complex numbers to the forefront, *Ars Magna* triggered the investigation of complex numbers.

Last, but not least, *Ars Magna* also motivated the investigation of equations of degree five and higher, culminating in the works of Abel and Galois. Thus, *Ars Magna* made significant contributions to the development of mathematics in general and algebra in particular. As stated by Cardano (1545), “Written in five years, may it last as many thousands” (p. 261).

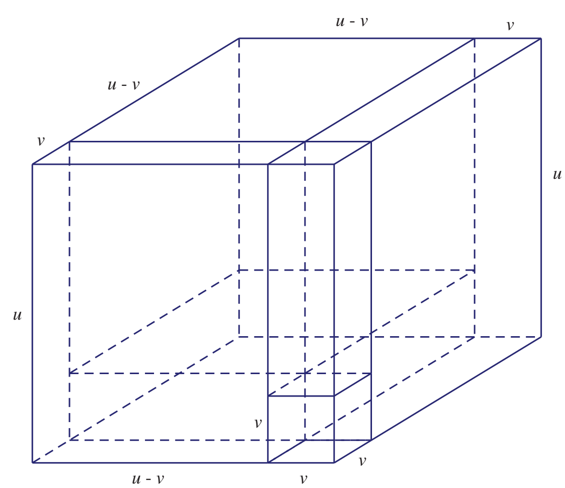

Figure 1: Cardano’s Cube

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