Variations of the COG Defuzzification Technique for Assessment Processes

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Abstract

In this paper two new fuzzy assessment models are developed, the trapezoidal (TRFAM) and the Triangular (TFAM), which are proved to be equivalent to each other. These models are new original variations of the Centre of Gravity (COG) defuzzification technique, which has been properly adapted and used in earlier papers as an assessment method. The central idea of TRFAM is the replacement of the rectangles appearing in the graph of the membership function of the COG technique by isosceles trapezoids sharing common parts. In this way one treats better the ambiguous cases being at the boundaries between two successive linguistic assessment characterizations (grades). The corresponding idea of TFAM is the replacement of the rectangles by isosceles triangles. Two applications are also presented (students' and Bridge players' assessment) illustrating our results. In these applications the TRFAM and TFAM are validated through their comparison with other, already established, assessment methods.

1. Introduction

The assessment of a system's effectiveness (i.e. of the degree of attainment of its targets) with respect to an action performed within the system (e.g. problem-solving, decision making, learning performance, etc) is a very important task that enables the correction of the system's weaknesses resulting to the improvement of its general performance.

The several methods in use for assessing a system's performance focus on different targets: Some of them measure the *mean system's performance* (e.g. the calculation of the mean of the scores obtained by a group of individuals), while others focus to its *quality performance* by assigning greater coefficients (weights) to the higher scores (e.g. the widely used in the USA Grade Point Average Index). Therefore, one who wants to obtain a comprehensive view of a system's performance must make use of more than one assessment methods for this purpose.

The assessment methods that are commonly used in practice are based on the principles of the classical, bivalent logic (yes-no). However, there are cases where a crisp characterization is not probably the proper one for the assessment. For example, a teacher is frequently not sure about a particular numerical grade characterizing a student's performance. Fuzzy logic, due to its nature of characterizing a case with multiple values, offers wider and richer resources covering such kind of cases.

In this paper we develop two new original fuzzy assessment models, which are equivalent to each other: The *Triangular Fuzzy Assessment Model* (TFAM) and the *Trapezoidal Fuzzy Assessment Model* (TRFAM). These models are variations of the very popular in fuzzy mathematics *Centre of Gravity (COG) defuzzification technique*, which we have properly adapted in earlier papers and used it as a general assessment method of a system's performance. TFAM and TRFAM are treating better than COG the ambiguous cases being at the boundaries between two successive assessment grades. Two real life applications (students' and Bridge players' assessment) are also presented illustrating our results in practice.

For general facts on fuzzy sets we refer to the book of Klir and Folger [4]

2. Background

There used to be a tradition in science and engineering of turning to probability theory when one is faced with a problem in which uncertainty plays a significant role. This transition was justified when there were no alternative tools for dealing with the uncertainty. Today this is no longer the case. *Fuzzy logic*, which is based on fuzzy sets theory introduced by Zadeh [22] in 1965, provides a rich and meaningful addition to standard logic. The applications which may be generated from or adapted to fuzzy logic are wide-ranging and provide the opportunity for modelling under conditions which are inherently imprecisely defined, despite the concerns of classical logicians.

Fuzzy Logic, due to its property of characterizing the ambiguous cases of a phenomenon by multiple values, has been widely used recently to solve problems in the evaluation tasks (e.g. [6, 7, 10 18, 19, 20], etc) In earlier works we have utilized the corresponding system's total uncertainty as a measurement of its performance (e.g. [18, 19], etc). In fact, as it is well known from the classical Information Theory [9], the reduction of a system's uncertainty as a result of an action performed within the system is connected to the information obtained by this action. Consequently, the lower is the system's uncertainty after the action, the greater is the amount of information obtained by the action. In other words, the system's effectiveness with respect to this action can be measured by the amount of its total uncertainty. This assessment method is connected to the system's mean performance. On the contrary, the COG defuzzification technique has been adapted and used in earlier papers (e.g. [10, 19, 20], etc) as a general assessment method of a system's quality performance. Below we shall sketch the above two fuzzy assessment methods, because we are going to use them in our applications, together with the new TRFAM and TFAM models.

3. Measuring the Uncertainty

According to the standard probability theory a system's uncertainty (and the information connected to it) is measured by the Shannon's formula which is known as the *Shannon's entropy* [9]. For use in a fuzzy environment the above formula has been expressed in the form: H= $-\frac{1}{\ln n}\sum_{s=1}^{n} m_s \ln m_s$ ([5], p. 20), where U is the universal set of the discourse, m: U $\rightarrow$ [0, 1] is the membership function of the corresponding fuzzy set, $m_s = m(s)$ denotes the membership degree of the element s of U and n denotes the total number of the elements of U. In the above formula the sum is divided by the natural logarithm of n in order to be normalized. Thus H takes values within the real interval [0, 1].

We recall that the *fuzzy probability* of an element s of U is defined in a way analogous to the crisp probability, i.e. by $P_s = \frac{m_s}{\sum_{s\in U} m_s}$. However, according to Shackle [8] and many other researchers after him, human reasoning can be formulated more adequately by the possibility rather, than by the probability theory. The *possibility* r_s of s is defined by $r_s = \frac{m_s}{\max\{m_s\}}$, where max $\{m_s\}$ denotes the maximal value of m_s, for all s in U. In other words, the possibility of s expresses the relative membership degree of s with respect to max $\{m_s\}$.

Within the domain of possibility theory uncertainty consists of *strife* or *discord*, which expresses conflicts among the various sets of alternatives, and *non-specificity* or *imprecision*, which indicates that some alternatives are left unspecified, i.e. it expresses conflicts among the cardinalities of the various sets of alternatives ([5], p.28). For a better intuitive understanding of the above two types of uncertainty we present the following simple example:

EXAMPLE: Let U be the set of integers from 0 to 130 representing the humans' ages and let Y = young, A = adult and O = old be fuzzy subsets of U defined by the membership functions m_Y, m_A and m_O respectively, where people are considered as young, adult or old according to their outer appearance . Then, given x in U, there usually exists a degree of uncertainty about the reasonable values that the membership degrees $m_Y(x)$, $m_A(x)$ and $m_O(x)$ could take, resulting to a conflict among the fuzzy subsets Y, A and O of U. For instance, if x = 18, values like $m_Y(x)$ = 0.8 and $m_A(x)$ = 0.3 are acceptable, but they are not the only ones. In fact, the values $m_Y(x)$ = 1 and $m_A(x)$ = 0.5 are also acceptable, etc. The existing conflict becomes even greater if x =50. In fact, is it reasonable in this case to take $m_Y(x)$ =0? Probably not, because sometimes people being 50 years old look much younger than others aged 40 or even 30 years. But, there exist also people aged 50 who look older from others aged 70, or even 80 years! So what about the acceptable values of $m_O(x)$? All the above are examples of the type of uncertainty that we have termed as strife. On the other hand, non - specificity is connected to the question: How many x in U should have non zero membership degrees in Y, A and O respectively? In other words, the existing in this case uncertainty creates a conflict among the cardinalities (sizes) of the fuzzy subsets of U. We recall that the *cardinality* of a fuzzy subset, say B, of U is defined to be the sum $\sum_{x \in U} m_B(x)$ of all membership degrees of the elements of U in B. -

Strife is measured by the function $ST(r)$ on the ordered possibility distribution r: $r_1=1 \geq r_2 \geq \ldots\ldots \geq r_n \geq r_{n+1}$ of the elements of U with respect to the corresponding fuzzy subset of U defined by $ST(r) = \frac{1}{\log 2}[\sum_{i=2}^{m}(r_i - r_{i+1})\log\frac{i}{\sum_{j=1}^{i} r_j}]$ ([5], p.28).

Similarly non-specificity is measured by the function $N(r) = \frac{1}{\log 2}[\sum_{i=2}^{m}(r_i - r_{i+1})\log i]$ ([5], p.28).

The sum $T(r) = ST(r) + N(r)$ measures the *total possibilistic uncertainty* for ordered possibility distributions. The lower is the value of $T(r)$, which means greater reduction of the initially (before the action) existing uncertainty, the better is the system's performance with respect to this action.

4. Use of the COG technique as an assessment method

For applying in fuzzy mathematics the COG defuzzification technique we correspond to each x of the universal set U an interval of values from a prefixed numerical distribution, which actually means that we replace U with a set of real intervals. Then, we construct the graph of the corresponding membership function $y=m(x)$. There is a commonly used in fuzzy logic approach (e.g. see [17]) to represent the fuzzy data with the pair of numbers (x_c, y_c) as the coordinates of the COG, say F_c, of the level's section S contained between the above graph and the OX axis, which we can calculate using the following well-known from Mechanics formulas:

$$x_c = \frac{\iint_S x\,dxdy}{\iint_S dxdy}, \quad y_c = \frac{\iint_S y\,dxdy}{\iint_S dxdy} \qquad (1).$$

In earlier papers Subbotin and Voskoglou have properly adapted the COG technique and used it as an assessment method ([10, 19, 20], etc). In fact, let G be a group of individuals participating in a certain activity and let U={A, B, C, D, F} be a set of linguistic labels (grades) characterizing the individuals' performance with respect to this activity as follows: A=excellent, B=very good, C=good, D= fair and F= unsatisfactory. Then, we can express G as a *fuzzy set* in U in the form G = {(x, $m(x)$), $x \in$ U}, where $y=m(x)$ is the corresponding

membership function.
We correspond to each x in U an interval of real values as follows: F $\rightarrow$ [0, 1), D $\rightarrow$ [1, 2), C $\rightarrow$ [2, 3),
B $\rightarrow$ [3, 4), A $\rightarrow$ [4, 5]. Consequently, we have that $y_1 = m(x) = m(F)$ for all x in [0,1), $y_2 = m(x) = m(D)$ for all x in [1,2), $y_3 = m(x) = m(C)$ for all x in [2, 3), $y_4 = m(x) = m(B)$ for all x in [3, 4) and $y_5 = m(x) = m(A)$ for all x in [4,5]. Then the graph of the membership function $y = m(x)$, takes the form of the bar graph of Figure 1, while the area of the level's section S contained between this graph and the OX axis is equal to the sum of the areas of the rectangles S_i, i=1, 2, 3, 4, 5.

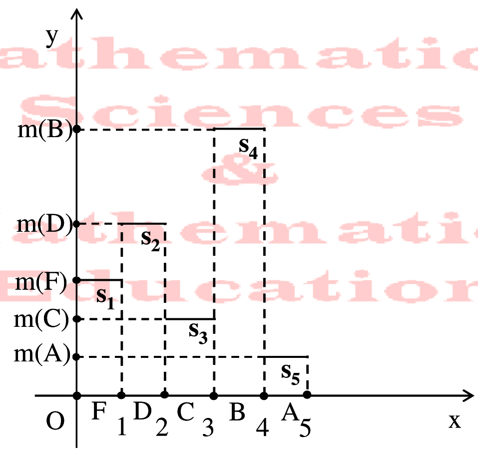

Figure 1: *Bar graphical data representation*

It is straightforward then to check that in this case formulas (1) are transformed to the form:

$$x_c = \frac{1}{2}\left(\frac{y_1 + 3y_2 + 5y_3 + 7y_4 + 9y_5}{y_1 + y_2 + y_3 + y_4 + y_5}\right),\ y_c = \frac{1}{2}\left(\frac{y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2}{y_1 + y_2 + y_3 + y_4 + y_5}\right) \tag{2}$$

with $y_i = m(x_i)$, i=1, 2, 3, 4, 5 and x_1=F, x_2=D, x_3=C, x_4=B and x_5=A.

In fact, $\iint_S dxdy$ is the area of S which is equal to $\sum_{i=1}^{5} y_i$, Also $\iint_S xdxdy$

$$= \sum_{i=1}^{5} \iint_{S_i} xdxdy = \sum_{i=1}^{5} \int_0^{y_i} dy \int_{i-1}^{i} xdx$$

$$= \sum_{i=1}^{5} y_i \int_{i-1}^{i} x dx = \sum_{i=1}^{5} y_i [\frac{x^2}{2}]_{i-1}^{i} = \frac{1}{2}\sum_{i=1}^{5} y_i [i^2 - (i-1)^2] = \frac{1}{2}\sum_{i=1}^{5} (2i-1) y_i$$

and $\iint_S y dx dy =$

$$\sum_{i=1}^{5} \iint_{F_i} y dx dy = \sum_{i=1}^{5} \int_{0}^{y_i} y dy \int_{i-1}^{i} dx = \sum_{i=1}^{n} \int_{0}^{y_i} y dy = \frac{1}{2}\sum_{i=1}^{n} y_i^{2}$$

Normalizing the membership degrees by dividing each y_i by the sum $\sum_{i=1}^{5} y_i$ we may assume without loss of generality that $\sum_{i=1}^{5} y_i = 1$. Therefore formulas (2) can be finally written in the form:

$x_c = \frac{1}{2}(y_1+3y_2+5y_3+7y_4+9y_5)$, $y_c = \frac{1}{2}(y_1^2+y_2^2+y_3^2+y_4^2+y_5^2)$ (3) ,

with $y_i = \frac{m(x_i)}{\sum_{j=1}^{5} m(x_j)}$, where x_1=F, x_2=D, x_3=C, x_4=B and x_5=A

Next, using elementary algebraic inequalities it is easy to check that there is a unique minimum for y_c corresponding to COG $F_m(\frac{5}{2}, \frac{1}{10})$ (e.g. [19], section 3, pp. 232-233). Further, the ideal case is when $y_1=y_2=y_3=y_4=0$ and $y_5=1$. Then from formulas (3) we get that $x_c = \frac{9}{2}$ and $y_c = \frac{1}{2}$. Therefore the COG in this case is the point $F_i(\frac{9}{2}, \frac{1}{2})$. On the other hand the worst case is when $y_1=1$ and $y_2=y_3=y_4= y_5=0$. Then from formulas (3) we find that the COG is the point $F_w(\frac{1}{2}, \frac{1}{2})$. Therefore the COG F_c of the level's section S lies in the area of the triangle $F_wF_mF_i$.

Then by elementary geometric observations (e.g. [19], section 3, 233) one can obtain the following criterion:

- *Between two groups the group with the bigger x_c performs better.*
- *If the two groups have the same $x_c \geq 2.5$, then the group with the bigger y_c performs better.*
- *If the two groups have the same $x_c < 2.5$, then the group with the lower y_c performs better.*

As it becomes evident by the above description, the application of the COG method is simple in its final application, because in contrast to the measurement of the system's total uncertainty [18], needs no complicated calculations in its final step. However, we must emphasize that the COG method treats differently the idea of a system's performance, than the measurement of the uncertainty does. In fact, as it can be easily observed by the above criterion and the first of formulas (3), the weighted average plays the main role in the COG method, i.e. the result of the system's performance close to its ideal performance has much more "weight" than the one close to the lower end. In other words, the COG method focuses on the systems quality performance.

5. Main Focus of the Paper

In this section we develop the new original TRFAM and the TFAM models and we show that they are equivalent to each other.

5.1 The Trapezoidal Fuzzy Assessment Model (TRFAM)

The TRFAM is a recently developed [13. 14] variation of the COG method presented in the previous section. The novelty of this approach is in the replacement of the rectangles appearing in the graph of the membership function of the COG method (Figure 1) by isosceles trapezoids sharing common parts, so that to cover the ambiguous cases of individuals' scores being at the boundaries between two successive grades. In the TRFAM's scheme (Figure 2) we have five trapezoids, corresponding to the individuals' performance characterizations F, D, C, B and A respectively defined in the previous section. Without loss of generality and for making our calculations easier we consider isosceles trapezoids with bases of length 10 units lying on the OX axis. The height of each trapezoid is equal to the percentage of individuals who achieved the corresponding characterization for their performance, while the parallel to its base side is equal to 4 units.

We allow for any two adjacent trapezoids to have 30% of their bases (3 units) belonging to both of them. In this way we treat better the ambiguous cases of individuals' scores being at the boundaries between two successive grades. For example, in students' assessment it is a very common approach to divide the interval of the specific grades in three parts and to assign the corresponding grade using + and - . For example, we could have 75 – 77 = B-, 78 – 81 = B, 82 – 84 = B+. However, this consideration does not reflect the common situation, where the teacher is not sure about the grading of the students whose performance could be assessed as marginal between and close to two adjacent grades; for example, something like 84 - 85 being between B_+ and A-. The TRFAM fits better than the COG technique to this kind of situations.

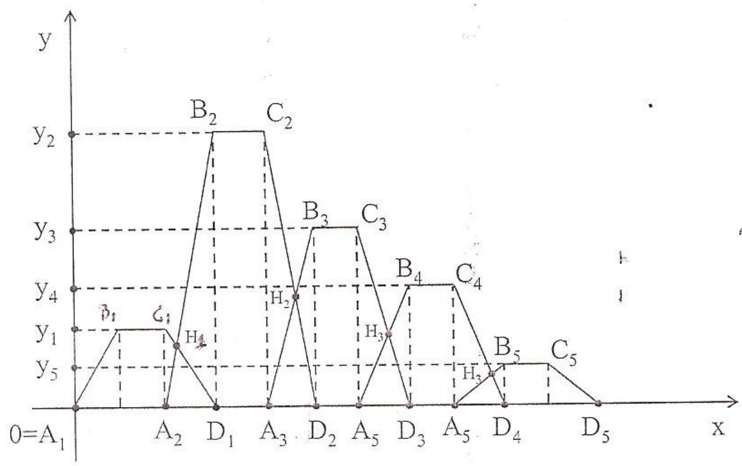

Figure 2: *The TRAFM's scheme*

In TRFAM an individuals' group can be represented, as in the COG method, as a fuzzy set in U, whose membership function $y=m(x)$ has as graph the line $OB_1C_1H_1B_2C_2H_2B_3C_3H_3B_4C_4H_4B_5C_5D_5$ of Figure 2, which is the union of the line segments OB_1, B_1C_1, C_1H_1,…….., B_5C_5, C_5D_5. However, in case of the TRFAM the analytic form of $y = m(x)$ is not needed for calculating the COG of the resulting area. In fact, since the marginal cases of the individuals' scores are considered as common parts for any pair of the adjacent trapezoids, it is logical to count these parts twice; e.g. placing the ambiguous cases B+ and A- in both regions B and A. In other words, the COG technique, which calculates the coordinates of the COG of the area between the graph of the membership function and the OX axis, thus considering the areas of the "common" triangles $A_2H_1D_1$, $A_3H_2D_2$, $A_4H_3D_3$ and $A_5H_4D_4$ only once, is not the proper method to be applied in the above situation.

Instead, in this case we represent each one of the five trapezoids of Figure 2 by its COG F_i, i=1, 2, 3, 4, 5 and we consider the entire area, i.e. the sum of the areas of the five trapezoids, as the system of these points-centers. More explicitly, the steps of the whole construction of the TRFAM are the following:

1. Let y_i, i=1, 2, 3, 4, 5 be the percentages of students whose performance was characterized by F, D, C, B, and A respectively; then $\sum_{i=1}^{5} y_i$ =1 (100%).

2. We consider the isosceles trapezoids with heights being equal to y_i, i=1, 2, 3, 4, 5, in the way that has been illustrated in Figure 2.

3. We calculate the coordinates (x_{c_i} , y_{c_i}) of the COG F_i, i=1, 2, 3, 4, 5, of each trapezoid as follows: It is well known that the COG of a trapezoid lies

along the line segment joining the midpoints of its parallel sides a and b at a distance d from the longer side b given by $d=\frac{h(2a+b)}{3(a+b)}$, where h is its height (e.g. see [23]).Therefore in our case we have $y_{c_i} = \ = \frac{y_i(2*4+10)}{3*(4+10)} = \frac{3y_i}{7}$. Also, since the abscissa of the COG of each trapezoid is equal to the abscissa of the midpoint of its base, it is easy to observe that $x_{ci}=7i-2$.

4. We consider the system of the COG's F_i, i=1, 2, 3, 4, 5 and we calculate the coordinates (X_c, Y_c) of the COG F_c of the whole area S considered in Figure 2 by the following formulas, derived from the commonly used in such cases definition (e.g. see [24]):

$$X_c=\frac{1}{S}\sum_{i=1}^{5} S_i x_{c_i}\,,\ Y_c=\frac{1}{S}\sum_{i=1}^{5} S_i y_{c_i} \quad (4).$$

In formulas (4) Si, i= 1, 2, 3, 4, 5 denotes the area of the corresponding trapezoid. Thus, $Si=\frac{(4+10)y_i}{2}=7y_i$ and $S=\sum_{i=1}^{5} S_i = 7\sum_{i=1}^{5} y_i = 7$. Therefore, from formulas (4) we finally get that

$$X_c=\frac{1}{7}\sum_{i=1}^{5} 7y_i(7i-2)=(7\sum_{i=1}^{5} iy_i)\ -2\,,\ Y_c=\frac{1}{7}\sum_{i=1}^{5} 7y_i(\frac{3}{7}y_i)=\frac{3}{7}\sum_{i=1}^{5} y_i^{2} \quad (5).$$

.

5. We determine the area where the COG F_c lies as follows: For i, j=1, 2, 3, 4, 5, we have that $0\leq(y_i - y_j)^2=y_i^2+y_j^2-2y_iy_j$, therefore $y_i^2+y_j^2 \geq 2y_iy_j$, with the equality holding if, and only if, $y_i=y_j$. Therefore $1=(\sum_{i=1}^{5} y_i)^2= \sum_{i=1}^{5} y_i^2 + 2\sum_{\substack{i,j=1,\\ i\neq j}}^{5} y_i y_j \leq \sum_{i=1}^{5} y_i^2 + +2\sum_{\substack{i,j=1,\\ i\neq j}}^{5} (y_i^2+y_j^2) = 5\sum_{i=1}^{5} y_i^2$ or $\sum_{i=1}^{5} y_i^2 \geq \frac{1}{5}$ (6), with the equality holding if, and only if, $y_1 = y_2 = y_3 = y_4 = y_5 = \frac{1}{5}$. In the case of equality the first of formulas (5) gives that $X_c = 7(\frac{1}{5} + \frac{2}{5} + \frac{3}{5} + \frac{4}{5} + \frac{5}{5}) - 2 =$ 19. Further, combining the inequality (6) with the second of formulas (5) one finds that $Y_c \geq \frac{3}{35}$ Therefore the unique minimum for Y_c corresponds to the COG $F_m(19, \frac{3}{35})$. The ideal case is when $y_1=y_2=y_3= y_4=0$ and $y_5=1$. Then from formulas (5) we get that X_c = 33 and $Y_c = \frac{3}{7}$. Therefore the COG in this case is

the point $F_i\,(33, \frac{3}{7})$. On the other hand, the worst case is when $y_1=1$ and $y_2= y_3 = y_4= y_5=0$. Then from formulas (5), we find that the COG is the point $F_w(5, \frac{3}{7})$. Therefore the area where the COG F_c lies is the area of the triangle $F_w\,F_m\,F_i$ (see Figure 3).

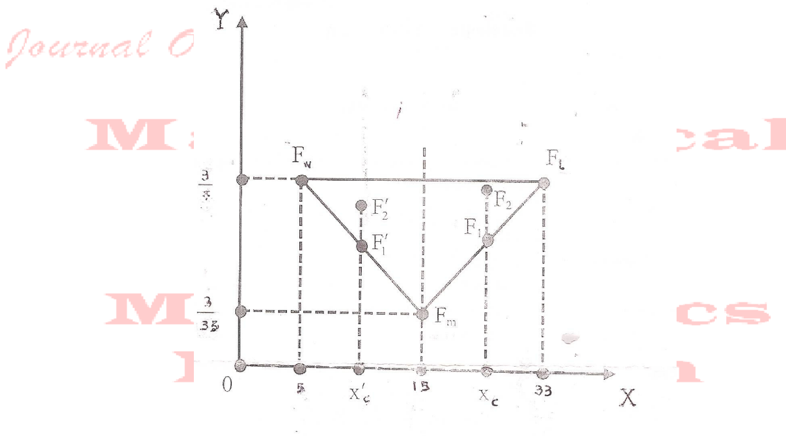

Figure 3: *The area where the COG lies*

6. We formulate our criterion for comparing the performances of two (or more) different student groups' as follows: From elementary geometric observations (see Figure 3) it follows that for two groups the group having the greater X_c performs better. Further, if the two groups have the same $X_c \geq 19$, then the group having the COG which is situated closer to *Fi* is the group with the greater Y_c. Also, if the two groups have the same $X_c<19$, then the group having the COG which is situated farther to *Fw* is the group with the smaller Y_c. Based on the above considerations it is logical to formulate our criterion for comparing the two groups' performance in the following form:

- *Between two groups the group with the greater value of X_c demonstrates the better performance.*
- *If two groups have the same $X_c \geq 19$, then the group with the greater value of Y_c demonstrates the better performance.*
- *If two groups have the same $X_c < 19$, then the group with the smaller value of Y_c demonstrates the better performance.*

The above criterion combined with the first of formulas (5) shows that the TRFAM measures the system's quality performance by assigning higher coefficients to the greater scores.

5.2 The Triangular Fuzzy Assessment Model (TFAM)

An equivalent to the TRFAM approach is to consider isosceles triangles instead of trapezoids ([13, 15, 21]. In this case we call the resulting framework *Triangular Fuzzy Assessment Model* (TFAM). The corresponding scheme is that shown in Figure 4.

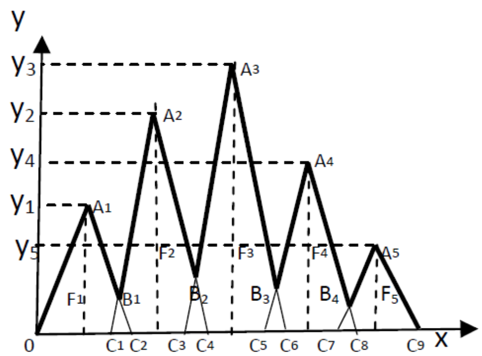

Figure 4: *The membership function's graph of TFAM*

For developing the TFAM we apply a similar argument as for the TRAFM above:

1. Let y_1, y_2 , y_3, y_4, y_5 be the percentages of the students in the group getting F, D, C, B, and A grades respectively, then $y_1 + y_2 + y_3 + y_4 + y_5 = 1$ (100%).

2. We consider the isosceles triangles with bases having lengths of 10 units each and their heights y_1, y_2, y_3, y_4, y_5 in the way that has been illustrated in Figure 1. Each pair of adjacent triangles has common parts in the base with length 3 units.

3.We calculate the coordinates (x_{c_i}, y_{c_i}) of the COG F_i, i=1, 2, 3, 4, 5 of each triangle as follows: The COG of a triangle is the point of intersection of its medians, and since this point divides the median in proportion 2:1 from the vertex, we find, taking also into account that the triangles are isosceles, that $y_{c_i} = \frac{1}{3} y_i$. Further, since the triangles' bases have a length of 10 units, it is easy to observe that $x_{ci} = 7i-2$.

4. We consider the system of the centers F_i, i=1, 2, 3, 4, 5 and we calculate the coordinates (X_c, Y_c) of the COG F_c of the whole level's area considered in

Figure 4 from formulas (4), where $Si = 5y_i$ and $S=\sum_{i=1}^{5} S_i = 5\sum_{i=1}^{5} y_i = 5$. Thus, one finds that the coordinates of the COG of the resulting in this case scheme are calculated by the formulas $X_c = (7\sum_{i=1}^{5} iy_i) - 2$, $Y_c = \frac{1}{5}\sum_{i=1}^{5} y_i^2$ (7).

Finally, working as in the above paragraphs 5 and 6 for TRFAM we obtain the same criterion for comparing the performance of two (or more) different group of individuals'.

As it can be easily observed from formulas (5) and (7) the only difference between the TRFAM and the TFAM concerns the value of the coordinate Y_c of the corresponding COG, but this does not affect the assessment results. Therefore, using the one or the other model makes no difference.

6. Applications

6.1 Students' Assessment

The students of two different Departments of the School of Management and Economics of the Graduate Technological Educational Institute of Western Greece achieved the following scores (in a climax from 0 to 100) at their common progress exam in the course "Mathematics for Economists I":

Department 1 (D_1): 100(5 times), 99(3), 98(10), 95(15), 94(12), 93(1), 92 (8), 90(6), 89(3), 88(7), 85(13), 82(4), 80(6), 79(1), 78(1), 76(2), 75(3), 74(3), 73(1), 72(5), 70(4), 68(2), 63(2), 60(3), 59(5), 58(1), 57(2), 56(3), 55(4), 54(2), 53(1), 52(2), 51(2), 50(8), 48(7), 45(8), 42(1), 40(3), 35(1).

Department 2 (D_2) *:* 100(7), 99(2), 98(3), 97(9), 95(18), 92(11), 91(4), 90(6), 88(12), 85(36), 82(8), 80(19), 78(9), 75(6), 70(17), 64(12), 60(16), 58(19), 56(3), 55(6), 50(17), 45(9), 40(6).

The linguistic characterizations (grades) mentioned in section 3 were assigned to the above scores as follows: A (100-85), B (84-75), C (60-74), D(50-59) and F (<50). The students' results with respect to the above grades are summarized in Table 1.

Table 1: *Characterization of the students' performance*

Characterizations	D_1	D_2
A	60	60
B	40	90
C	20	45
D	30	45
E	20	15
Total	170	255

In order to check the effectiveness of the fuzzy assessment methods presented in this paper, the evaluation of the above data will be performed in two ways: I) By two very common traditional assessment methods based on principles of the bivalent logic (yes-no) and II) by applying our fuzzy methods. Then the results obtained will be compared and the proper conclusions will be drawn.

i) Calculation of the means: A straightforward calculation gives that the means of the above presented students' scores are approximately equal to 76.006 and 75.09 for D_1 and D_2 respectively. This shows that the *mean performance* of both student groups was very good (on the boundary), with the performance of the group D_1 being slightly better.

(ii) Calculation of the GPA index: We recall that the *Grade Point Average* (*GPA*) is a weighted mean, where more importance is given to the higher scores achieved, to which greater coefficients (weights) are attached. In other words, the GPA method focuses on the *quality performance* of a student group. For applying the GPA method on the data of our experiment let us denote by n_A, n_B, n_C, n_D and n_F the numbers of students whose performance was characterized by A, B, C, D and F respectively and by n the total number of students of each group. .Then the GPA index is calculated by the formula GPA=$\frac{n_D + 2n_C + 3n_B + 4n_A}{n}$. Using the notation of section 4 the above formula can be written in the form GPA= $y_2+2y_3+3y_4+4y_5$ (8).

It is easy to observe that $0 \leq \text{GPA} \leq 4$. In fact, GPA=0, if $n_F = n$ (worst case), while GPA=4, if $n_A = n$ (ideal case)

In our case, applying formula (8) on the data of Table 1 one finds that the GPA of both student groups' is equal to $\frac{43}{17} \approx 2.529$. Thus, the two student groups demonstrated the same quality performance. Further, their performance can be characterized as satisfactory, since the value 2.529 of the GPA index is greater than the half of its maximal possible value, which is equal to 4.

(iii) Measurement of the uncertainty: We represent the two student groups as fuzzy sets in U. For this, we define the membership function $m: U \rightarrow [0, 1]$ for both groups D_1 and D_2 .by $y = m(x) = \frac{n_x}{n}$, for all x in U, where the notation for n_x is the same as in the above case (ii) of the GPA index. Then, from Table 1 it turns easily out that D_1 and D_2 can be written as fuzzy sets in U in the form

$D_1 = \{(A, \frac{6}{17}), (B, \frac{4}{17}), (C, \frac{2}{17}), (D, \frac{3}{17}), (F, \frac{2}{17})\}$ and

$D_2 = \{(A, \frac{4}{17}), (B, \frac{6}{17}), (C, \frac{3}{17}), (D, \frac{3}{17}), (F, \frac{1}{17})\}$ respectively.

From Table 1 we find that *max* $\{m_x\} = \frac{6}{17}$ for both groups, therefore the possibilities of the elements of U are calculated by the formula $r_s = \frac{m_s}{\frac{6}{17}}$ for both groups. Performing the corresponding calculations we find that $r_1=1$, $r_2=\frac{2}{3}$, $r_3=\frac{1}{2}$, $r_4=r_5=\frac{1}{3}$ for D_1 and $r_1=1$, $r_2=\frac{2}{3}$, $r_3=r_4=\frac{1}{2}$, $r_5=\frac{1}{6}$ for D_2. Replacing the above values of possibilities to the corresponding formulas of section 2 we find for D_1 that

$$S(r)=\frac{1}{\log 2}[\sum_{i=2}^{4}(r_i - r_{i+1})\log\frac{i}{\sum_{j=1}^{i} r_j}] = \frac{1}{\log 2}(\frac{1}{6}\log\frac{6}{5}+\frac{1}{6}\log\frac{18}{13}) \approx 0.043,$$

$$N(r)=\frac{1}{\log 2}[\sum_{i=2}^{4}(r_i - r_{i+1})\log i] = \frac{1}{\log 2}(\frac{1}{6}\log 2+\frac{1}{6}\log 3) \approx 0.431$$ and

$T(r) \approx 0.043+0.431=0.474$.

Similarly we find for D_2 that $S(r)=\frac{1}{\log 2}(\frac{1}{6}\log\frac{6}{5}+\frac{2}{6}\log\frac{12}{8}) \approx 0.239$,

$N(r) = \frac{1}{\log 2}(\frac{1}{6}\log 2+\frac{2}{6}\log 4) \approx 0.695$ and $T(r) \approx 0.239+0.695 = 0.934$.

Therefore D_1 demonstrates a considerably better performance than D_2.

(iv) The COG technique: In our application we have $\sum_{i=1}^{5} y_i$ = m(A) + m (B) + m(C) + m(D) + m(F) = $\frac{n_A+n_B+n_C+n_D+n_F}{n}$ =1. Therefore, replacing the values of y_i's taken from the fuzzy sets D_1 and D_2 of paragraph (iii) in the first of formulas (2) we find that the coordinate x_c of the COG for both D_1 and D_2 is equal to $\frac{103}{34} \approx 3.029 > 2.5$. Since the value 3.029 found for x_c is greater than the half of its value in the ideal case, which is $\frac{9}{2}$ (see section 3), both groups demonstrated a more than satisfactory performance. Further, by the second of formulas (2) one finds that the coordinate y_c of the COG is equal to $\frac{69}{578} \approx 0.119$ for D_1 and to $\frac{71}{578} \approx 0.122$ for D_2. Therefore, according to our criterion stated in section 3, D_2 demonstrated a slightly better performance than D_1.

(v) Application of the TRFAM: Replacing the values of y_i's in the first of formulas (5) we find that $X_c = \frac{386}{17} \approx 22.706 > 15$ for both groups. This means that the quality performance of both groups was more than satisfactory, since the value 22.706 is greater than the half of the value of X_c in the ideal case, which is equal to 33 (see section 4) Also, the second of formulas (5) gives that $Y_c = \frac{3}{7} * \frac{69}{286} \approx 0.103$ for D_1 and $Y_c = \frac{3}{7} * \frac{71}{286} \approx 0.106$ for D_2. Thus, according to the criterion stated in section 4, D_2 demonstrated a slightly better performance than D_1.

(vi) Application of the TFAM: Analogous results are obtained if, instead of formulas (5) for TRFAM, we apply formulas (7) of TFAM, the only difference being with the values of Y_c, which are approximately equal to 0.048 and 0.05 for D_1 and D_2 respectively.

(vii) Comparison of the Assessment Methods: The application of the above methods resulted to different conclusions. However, this is not embarrassing, since, in contrast to the calculation of the means and the measurement of the system's uncertainty, which focus on the mean performance of a student group, the GPA, the COG and the TRFAM methods focus on its quality performance by assigning weight coefficients to the higher scores achieved by students. This explains why, although D_1 demonstrated a better performance with respect to the calculation of the means and the measurement of the system's uncertainty, the performance of D_2 was found to be equal or better than the performance of D_1, when using the GPA the COG and the TRFAM methods

The coefficients attached to the y_i's in the last three methods -see formula (8) and the first of formulas (2) and (5) respectively- are presented in the following Table 2:

Table 2: *Weight coefficients of the y_i's*

y_i	GPA	COG (x_c)	TRAFM (X_c)
y_1	0	½	7
y_2	1	3/2	14
y_3	2	5/2	21
y_4	3	7/2	28
y_5	4	9/2	35

From Table 2 it becomes evident that the two fuzzy assessment methods (GOC and TRFAM) assign greater coefficients to the higher scores than GPA. In other words these two methods are *more sensitive* than GPA to the higher scores. This explains why the quality performance of the two groups was found to be the same with respect to the GPA index, while D_2 demonstrated a slightly better performance with respect to the COG technique and the TRFAM.

Notice also that, since the COG technique and the TRFAM treat differently the ambiguous cases of the students' scores being at the boundaries between two successive assessment grades, the conclusions obtained by applying these two assessment methods could differ in certain (other than the present) cases.

In concluding, the above performed comparison of our fuzzy methods with the two traditional assessment methods provided a very strong indication for their efficiency. Also, from the previous discussion it becomes evident that, although the proposed fuzzy assessment methods can be applied independently, the combined use of them gives to the user a more comprehensive view of the system's performance. On the other hand, if someone has personal criteria of goals (e.g. he/she is interested to the mean system's performance only), it is suggested to choose the method (or methods) that fits better to these criteria.

6.2 Assessing the Bridge Players' Performance

The *Contract Bridge* is a card game belonging to the family of trick-taking games. It occupies nowadays a position of great prestige being, together with chess, the only *mind sports* (i.e. games or skills where the mental component is more significant than the physical one) officially recognized by the International Olympic Committee. Millions of people play bridge worldwide in clubs, tournaments and championships, but also on line and with friends at home, making it one of the world's most popular card games.

A match of bridge can be played either among *teams* (two or more) of four players (two partnerships), or among *pairs*. For a pairs event a minimum of three tables (6 pairs, 12 players) is needed, but it works better with more players. At the end of the match in the former case the result is the difference in *International Match Points (IMPs)* between the competing teams and then there is a further conversion, in which some fixed number of *Victory Points (VPs)* is appointed between the teams. It is worthy to notice that the table converting IMPs to VPs has been obtained through a rigorous mathematical manipulation [1].

On the contrary, the usual method of scoring in a pairs' competition is in *match points*. Each pair is awarded two match points for each pair who scored worse than them on each game's session (*hand*), and one match point for each pair who scored equally. The total number of match points scored by each pair over all the hands played is calculated and it is converted to a percentage. However, IMPs can also used as a method of scoring in pair events. In this case the difference of each pair's IMPs is usually calculated with respect to the mean number of IMPs of all pairs.

For the fundamentals and the rules of bridge, as well as for the conventions usually played between the partners we refer to the famous book [3] of *Edgar Kaplan* (1925-1997), who was an American bridge player and one of the principal contributors to the game. Kaplan's book was translated in many languages and was reprinted many times since its first edition in 1964. There is also a fair amount of bridge-related information on the Internet.

The *Hellenic Bridge Federation (HBF)* organizes, on a regular basis, *simultaneous* bridge tournaments (pair events) with pre-dealt boards, played by the local clubs in several cities of Greece. Each of these tournaments consists of six in total events, played in a particular day of the week (e.g. Wednesday), for six successive weeks. In each of these events there is a local scoring table (match points) for each participating club, as well as a central scoring table, based on the local results of all participating clubs, which are compared to each other. At the end of the tournament it is also formed a total scoring table in each club, for each player individually. In this table each player's score equals to the mean of the scores obtained by him/her in the five of the six in total events of the tournament. If a player has participated in all the events, then his/her worst score is dropped out. On the contrary, if he/she has participated in less than five events, his/her name is not included in this table and no possible extra bonuses are awarded to him/her.

In case of a pairs' competition with match points as the scoring method and according to the usual standards of contract bridge, one can characterize the players' performance, according to the percentage of success, say p, achieved by them, as follows:

- Excellent (A), if $p > 65\%$.
- Very good (B), if $55\% < p \leq 65\%$.
- Good (C), if $48\% < p \leq 55\%$.
- Mediocre (D), if $40\% \leq p \leq 48\%$.
- 6Unsatisfactory (F), if $p < 40\%$.

Our application presented here is related to the total scoring table of the players of a bridge club of the city of Patras, who participated in at least five of the six in total events of a simultaneous tournament organized by the HBF, which ended on February 19, 2014 [2]. Nine men and five women players are included in this table, who obtained the following scores. Men: 57.22%, 54.77%, 54.77%, 54.35%, 54.08%, 50.82 %, 50.82%, 49.61%, 47.82%. Women: 59.48%, 54.08%, 53.45%, 53.45%, 47.39%. The above results give a mean percentage of approximately 52.696% for the men and 53.57% for the women players. Therefore the women demonstrated a slightly better mean performance than the men players, their difference being only 0.874%. The above results are summarized in Table 3

Table 3: *Total scoring of the men and women players*

% Scale	Performance	Men	Women
>65%	A	0	0
55-65%	B	1	1
48-55%	C	7	3
40-48%	D	1	1
<40%	F	0	0
Total		9	5

From Table 3 we find that the percentages for the men and women players are $y_1=0$, $y_2=\frac{1}{9}$, $y_3=\frac{7}{9}$, $y_4=\frac{1}{9}$, $y_5=0$ and $y_1=0$, $y_2=\frac{1}{5}$, $y_3=\frac{3}{5}$, $y_4=\frac{1}{5}$, $y_5=0$ respectively. Replacing these values in formula (8) we find that the GPA index is equal to 2 for both men and women players, who therefore demonstrate the same quality performance.

Further formulas (3) for the COG technique give that x_c = 2.5 for both men and women, but y_c=0.31 for the men and y_c=0.22 for the women players. Therefore, according to the corresponding criterion, the men demonstrate a slightly better performance than the women players (on the boundary, since the value 2.5 is a critical value between the last two cases of the criterion). The same conclusion is obtained by applying the TRFAM. In fact, in this case formulas (5) give that X_c=19 for both men and women, but $Y_c \approx 0.27$ for the men and $Y\text{c} \approx 0.19$ for the women players (the value 19 is again critical for the corresponding criterion)

Our new fuzzy assessment methods for the bridge players' performance can be used as a complement of the usual scoring methods of the game (match points or IMPs) in cases where one wants to compare (for statistical or other reasons) the overall performance of special groups of players (e.g. men and women, young and old players, players of two or more clubs participating in a big tournament, etc).

7. Discussion and conclusion

7.1 Future Research Directions

Our future research plans include the application of our new assessment models on more sectors of human activities and the comparison of them with the already established assessment methods in each of these sectors. These sectors may include other competitive games (e.g. chess), collective and individual sports, human cognition and learning, Artificial Intelligence, Biomedical

Sciences, Management and Economics, etc. In this way we shall obtain more solid conclusions about their applicability in practice, about their advantages and disadvantages with respect to the other assessment methods, etc.

7.2 Conclusion

Fuzzy Logic, due to its nature of characterizing the ambiguous cases with multiple values offers reach resources for the evaluation tasks. In this chapter two original and equivalent to each other fuzzy assessment models were developed. These models are variations of the COG defuzzification technique, which has been properly adapted and used as an assessment method in earlier works. The main idea for the construction of these models is the replacement of the rectangles appearing in the graph of the COG technique by isosceles triangles (Triangular model) or trapezoids (Trapezoidal model) sharing common parts. In this way one treats better the ambiguous cases being at the boundaries between two successive assessment grades. The applicability and creditability of our new models was tested by applying them on students' and Bridge players' assessment and by comparing them with other already established assessment methods (calculation of the means and of the GPA index and measurement of the corresponding system's total possibilistic uncertainty).

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