# Discovering Patterns and Teaching Number Sense in a Technology-Based Exploratory Teacher Education Environment

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**Abstract**

The Tower of Hanoi is a classic mathematics problem utilized to teach number sense, investigate patterns, evaluate exponential expressions, generalize nonhomogeneous recurrence relations, and prove by mathematical induction in k-16 curriculum. Mathematics educators utilize and discuss some version of this problem in different level of mathematics classes.

The derivation of the story is that a group of monks were the guardians of 64 golden rings which were stacked on top of each other with larger rings beneath the smaller ones. The rings were to be moved one at the time from the first tower to the third tower. Larger rings were not allowed to be placed on top smaller one. The second tower was sacred enough that rings could be rested there before they were placed permanently in the third tower. It is said that the ultimate fate of the universe, the Big Crunch, happens as soon as the monks complete their task.

A simple version of the problem is taught in k-5 classes using manipulative. An investigated version of the problem is used in middle and high school mathematics to discover different types of patterns and evaluate exponential expressions. At the college level the tower of Hanoi is an excellent example to apply the principles of generalization of nonhomogeneous recurrence relations as well as, to prove by mathematical induction. We employed a novel method by utilizing the Excel software as a learning-teaching strategy to investigate the tower of Hanoi and calculate the minimum number of moves without complicated calculations and mathematical proof in a pre-service teacher environment.

Our quest was to encourage pre-service teachers to utilize relevant technology and provides new and richer contexts for teaching and learning of mathematical concepts. Student's written journals and portfolios indicated that they were highly motivated, interested, engaged, persevered, and made connections during this activity.



## 1 Background

The Tower of Hanoi (named after a city in Vietnam) or Tower of Brahma (named after a temple in the Indian city of Benares) is a mathematics game or problem well known to students of mathematics and computer science since. Legend goes that there was a temple in Hanoi which marked the center of the universe. There were three towers in the temple. A group of monks were the guardians of 64 golden rings which were stacked in the first tower with each ring smaller than the one beneath it. The monks were to move the rings from this first tower to the third tower one at a time. They were not allowed to place larger ring on top of a smaller one at any time. They must always move the top ring. It is said that as soon as the monks complete their task the temple will collapse and the universe will come to the end. The problem was introduced by the French mathematician Edouard Lucas in 1883. Since there is no acknowledged written documentation about the problem prior to that, it gives the impression that the puzzle probably was invented by Lucas.



The Tower of Hanoi is a problem often used to:

1. To discover patterns.
2. To evaluate an exponential expression.
3. To examine nonhomogeneous recurrence relations
4. To prove by mathematical induction

## 2 Methods and Procedures

Fifty nine students from two different sections of an algebra course designed pre-service teachers participated in this project. We began with the simpler version of the Tower of Hanoi problem. The students were asked to carry out the following activity and solve the problem. Suppose we have three trays A, B, and C. We place three coins in tray A, a quarter at the bottom, a nickel in the middle, and a penny on the top (see figure 1).

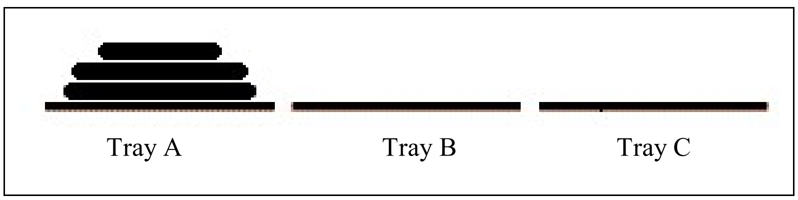


Figure 1: Manipulative for Tower of Hanoi

The objective is to move the coins from tray A to tray C one at a time. At no time a large coin is to be placed on top of a smaller coin. All coins may be placed temporarily in tray B to facilitate the procedure. We are interested to determine the minimum number of moves required to place the coins from tray A to tray C? The results are recorded in table I.

Table I: Minimum Number of Moves for 3 Coins

| Number of Coins | Minimum Number of Moves |
|---|---|
| 0 | 0 |
| 1 | 1 |
| 2 | 3 |
| 3 | 7 |

To press forward with more challenging version of the problem, we place 5 coins in tray A, a Dollar coin at the bottom, a half Dollar coin on top of it, a, quarter on top of half Dollar coin, a nickel on top of the quarter, and a penny on the very top. WE have the same objective and follow the same set of rules. The results are shown in table II.

Table II: Minimum Number of Moves for 5 Coins

| Number of Coins | Minimum Number of Moves |
|---|---|
| 0 | 0 |
| 1 | 1 |
| 2 | 3 |
| 3 | 7 |
| 4 | 15 |
| 5 | 31 |



We noticed that the number of the moves necessary to transfer one, two, three, four, or five disks follows a recursive pattern; this is a pattern that uses information from one step to find the next step. Table III demonstrate the existence of a recursive pattern.

Table III: The Existence of a Recursive Pattern.

| Number of Coins | Minimum Number of Moves | |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 1 | 2 (0)+1=1 |
| 2 | 3 | 2 (1)+1=3 |
| 3 | 7 | 2 (3)+1=7 |
| 4 | 15 | 2(7)+1=15 |
| 5 | 31 | 2 (15)+1=31 |

Revisiting the Tower of Hanoi problem, utilizing this procedure to identify the number of minimum moves to transfer 64 rings from tray A to tray C would be a lengthy process. First we have to find the minimum number of moves to transfer 63 rings, and before that the minimum number of moves to transfer 62 rings and so forth. Therefore, the recursive pattern will not be much help in finding the number of moves to transfer a large number of rings.

**Prove by Mathematical Induction:** In college level courses mathematics students proves by Mathematical Induction and show that the recurrence relation $a_n = 2a_{n-1} + 1$ will hold for all $n \geq 1$, and $a_0 = 0$ .

If $n = 1$ then,

$a_1 = 2a_0 + 1 = 2(0) + 1 = 1 = 2^1 - 1$, and therefore the base is true.

If $n = k$ then

$a_k = 2a_{k-1} + 1 = 2^k - 1$

Given that $k_{th}$ proposition is true we need to show that $n = k+1$ will hold. Since

$a_{k+1} = 2a_{k+1-1} + 1$

$a_{k+1} = 2a_{k+1-1} + 1 = 2a_k + 1 = 2(2^k - 1) + 1 = 2^{k+1} - 2 + 1 = 2^{k+1} - 1$

Therefore, $a_n = 2a_{n-1} + 1$ will hold for all $n \geq 1$



**Generalizing the Nonhomogeneous Recurrence Relation:** Grimaldi (1999) develops the following procedure to generalize the nonhomogeneous recurrence relation of the tower of Hanoi problem. The objective is to show that:

$a_{n+1} = 2a_n + 1$ where $n \geq 0$, and $a_0 = 0$

For $a_{n+1} - 2a_n = +1$, $a_n^{(h)} = c(2^n)$

Since $f(n) = 1 = (1)^n$ is not a solution of $a_{n+1} - 2a_n = 0$ we set

$a_n^{(p)} = A(1)^n = A$

We find from the given relation that $A = 2A + 1$ so that $A = -1$ and

$a_n = c(2^n) - 1$.

From $a_0 = 0 = c - 1$, we conclude that $c = 1$ and that $a_{n+1} = 2a_n + 1$ where $n \geq 0$

If we generalize the pattern to move n coins the minimum number of movement required is $2^n - 1$. This process is illustrated in Table IV.

Table IV: The Generalization of the Pattern

| Number of Coins | Minimum Number of Moves | | |
|---|---|---|---|
| 0 | 0 | 0 | $2^0 - 1 = 1 - 1 = 0$ |
| 1 | 1 | 2 (0)+1=1 | $2^1 - 1 = 2 - 1 = 1$ |
| 2 | 3 | 2 (1)+1=3 | $2^2 - 1 = 4 - 1 = 3$ |
| 3 | 7 | 2 (3)+1=7 | $2^3 - 1 = 8 - 1 = 7$ |
| 4 | 15 | 2(7)+1=15 | $2^4 - 1 = 16 - 1 = 15$ |
| 5 | 31 | 2 (15)+1=31 | $2^5 - 1 = 32 - 1 = 31$ |

**Utilizing the Excel software**: Abramovich and Norton (2000) state "Modern technology tools offer a dynamic environment in which to visualize a complexity of infinite processes". Employing the Excel will generate the number of moves to transfer as many rings as we wish. The minimum number of moves required to solve the tower of Hanoi for 64 rings is shown in Table III.

**Table III: The minimum number of moves required to solve the tower of Hanoi for 64 rings**



| Number of Coins | Minimum Number of Moves |
|---:|---:|
| 0 | 0 |
| 1 | 1 |
| 2 | 3 |
| 3 | 7 |
| 4 | 15 |
| 5 | 31 |
| 6 | 63 |
| 7 | 127 |
| 8 | 255 |
| 9 | 511 |
| 10 | 1023 |
| 11 | 2047 |
| 12 | 4095 |
| 13 | 8191 |
| … | … |
| 49 | $5.6295 \cdot 10^{14}$ |
| 50 | $1.1259 \cdot 10^{15}$ |
| 51 | $2.2518 \cdot 10^{15}$ |
| 52 | $4.5036 \cdot 10^{15}$ |
| 53 | $9.0072 \cdot 10^{15}$ |
| 54 | $1.80144 \cdot 10^{16}$ |
| 55 | $3.60288 \cdot 10^{16}$ |
| 56 | $7.20576 \cdot 10^{16}$ |
| 57 | $1.44115 \cdot 10^{17}$ |
| 58 | $2.8823 \cdot 10^{17}$ |
| 59 | $5.76461 \cdot 10^{17}$ |
| 60 | $1.15292 \cdot 10^{18}$ |
| 61 | $2.30584 \cdot 10^{18}$ |
| 62 | $4.61169 \cdot 10^{18}$ |
| 63 | $9.22337 \cdot 10^{18}$ |
| 64 | $1.84467 \cdot 10^{19}$ |

We asked the students to consider the Big Bang as the beginning of the time and the Big Crunch as the end of the time. According to the legend the ultimate fate of the universe, the Big Crunch, happens as soon as the monks complete their task. Furthermore, we asked the students to calculate the time necessary for the monks to transfer the 64 golden rings from the first tower to the third tower, at the rate of one move per second, if they operate 24 hours a day.

$1.84467\cdot 10^{19}$ Seconds

$$\frac{1.84467\cdot 10^{19}}{60} = 3.07445\cdot 10^{17} \text{ Minutes}$$

$$\frac{3.07445\cdot 10^{17}}{60} = 5.124083333\cdot 10^{15} \text{ Hours}$$

$$\frac{5.124083333\cdot 10^{15}}{24} = 2.135034722\cdot 10^{14} \text{ Days}$$

$$\frac{2.135034722\cdot 10^{14}}{365} = 5.84910198\cdot 10^{11} \text{ Years}$$

$$\frac{5.84910198\cdot 10^{11}}{10} = 5.84910198\cdot 10^{10} \text{ Decays}$$

$$\frac{5.84910198\cdot 10^{10}}{10} = 5.84910198\cdot 10^{9} \text{ Centuries}$$

$$\frac{5.84910198\cdot 10^{9}}{10} = 5.84910198\cdot 10^{8} = 584910198 \text{ Millenniums}$$



We reminded the students that Hawking, S. (1990) suggests that our universe is about ten or twenty billion years or $1\cdot 10^7$ to $2\cdot 10^7$ millenniums. We asked the students to assume that it is fifteen billion or $15\cdot 10^9 = 1.5\cdot 10^{10}$ years old. In addition, we asked them to assume that the monks began to transfer the golden rings from the first to the third tower shortly after the Big Bang! Furthermore, we asked them to calculate how much time is left!

$1.5\cdot 10^{10}$ Years = $1.5\cdot 10^{7}$ Millenniums.

$$5.84910198\cdot 10^8 - 1.5\cdot 10^7 = 5.84910198\cdot 10^8 - 0.15\cdot 108^8 = 4.69910198\cdot 10^8$$

$4.69910198\cdot 10^8 = 469910198$ Millenniums,

Therefore, we have a very long time left!

**3 Conclusion**

Our quest was to encourage pre-service teachers to utilize relevant technology and provides new and richer contexts for teaching and learning of mathematical concepts. The participants were highly motivated, interested, engaged, persevered, and made connections during this technology-enabled activity. They were encouraged not only to discover a pattern and find a relation between the number of coins (rings) and the least number of moves to transfer them from first tower to the third tower, but they were overwhelmed by employing a simple tool to generate large numbers. The participants exhibited perseverance in multiple trials to find the least number of moves.

Utilizing the power of technology channeled the participants into a self-directive role to place a greater focus on reasoning, understanding the numbers sense, reflecting, and connecting. We are in agreement with Ramirez O. et al. (2005), Pachnowski, L. and Jurczyk J.(2003) and trust that utilizing technology provides new and richer contexts for teaching and learning of mathematics.

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