Finding Extrema of Rational Quadratics without Calculus

Kenneth V. Cartwright, Ph.D. †

Abstract

Without using calculus, the formula for the x coordinates of the extrema of a ratio of quadratics is derived. This derivation uses a fact that has been known since at least the seventeenth century, i.e. each line of minimum or maximum intersects the rational quadratic at two identical points. Furthermore, a method to calculate the points of extrema without explicit use of the aforementioned formula is given. In fact, this alternative method first finds the y coordinates of the extrema and then the corresponding x coordinates, which is opposite to the method of calculus.

Introduction

Let $q(x)=\dfrac{ax^2+bx+c}{dx^2+ex+f}$ and (h,v) be a minimum or maximum point of $q(x).$ Then, $q(x)-v=0$ will have a double root at $x=h.$ This fact has been known since at least the seventeenth century [1], but has been rediscovered more recently in [2] and [3]. Hence, $q(x)-v=0$ has two identical solutions, i.e. $q(x)-v=(x-h)^2=0.$ This is illustrated in Fig. 1, which shows a plot of $q(x)=\dfrac{x^2+6x+3}{4x^2+5x+6}$ and the line of maximum $(y=0.672)$ and the line of minimum $(y=-0.503)$. Note that each of the lines of extrema intercepts the rational quadratic at exactly one point (more correctly, two identical points), i.e. each line is tangent to the rational quadratic curve [1]. All other lines of zero slope intercept the rational quadratic at two non-identical points, i.e. these lines are not tangent to the curve [1].

In this paper, this fact is used to derive an equation for the x coordinates of the extrema of $q(x),$ which of course is identical to the one provided by calculus, i.e.

$$h=\frac{-af+cd}{-bd+ae}\pm\sqrt{\left(\frac{af-cd}{-bd+ae}\right)^2-\left(\frac{fb-ce}{-bd+ae}\right)}. \qquad (1)$$

(Note the calculus derivation of (1) is given in the appendix).

The y coordinate is then found from $v = q(h) = \dfrac{ah^2 + bh + c}{dh^2 + eh + f}$.

Furthermore, an alternative method is presented to find the extrema without explicitly using (1). In fact, in this alternative method v is first found and then h.

Pre-Calculus Derivation of (1)

To begin, note that

$$\begin{aligned} &\frac{ax^2 + bx + c}{dx^2 + ex + f} - v = 0 \\ &\Rightarrow ax^2 + bx + c - v\left(dx^2 + ex + f\right) = 0 \\ &\Rightarrow \left(a - vd\right)x^2 + \left(b - ve\right)x + c - vf = 0 \\ &\Rightarrow x^2 + \frac{b - ve}{a - vd}x + \frac{c - vf}{a - vd} = 0. \end{aligned} \tag{2}$$

Solving (2) with the quadratic formula gives

$$x = -\frac{b - ve}{2\left(a - vd\right)} \pm \frac{1}{2}\sqrt{\left(\frac{b - ve}{a - vd}\right)^2 - 4\left(\frac{c - vf}{a - vd}\right)}. \tag{3}$$

However, in order for v to be an extremum, there must be two identical solutions to (2), i.e.

$$x = -\frac{b - ve}{2\left(a - vd\right)} \tag{4}$$

and

$$\left(\frac{b - ve}{a - vd}\right)^2 - 4\left(\frac{c - vf}{a - vd}\right) = 0. \tag{5}$$

Substituting (4) into (5) gives

$$x^2 - \left(\frac{c - vf}{a - vd}\right) = 0. \tag{6}$$

Solving (4) for v gives

$$a - vd = -\frac{b - ve}{2x}$$
$$\Rightarrow \left(\frac{e}{2x} + d\right) v = \frac{b}{2x} + a \qquad (7)$$
$$\Rightarrow v = \frac{\frac{b}{2x} + a}{\frac{e}{2x} + d} = \frac{b + 2ax}{e + 2dx}.$$

Substituting (7) into (6) produces

$$x^2 - \left(\frac{c - \frac{b+2ax}{e+2dx} f}{a - \frac{b+2ax}{e+2dx} d}\right) = 0$$
$$\Rightarrow x^2 - \left(\frac{c(e+2dx) - (b+2ax) f}{a(e+2dx) - (b+2ax) d}\right) = 0 \qquad (8)$$
$$\Rightarrow x^2 - \frac{(-2af + 2dc)x - bf + ce}{ae - bd} = 0$$
$$\Rightarrow x^2 + \frac{2af - 2dc}{ae - bd} x + \frac{-bf + ce}{ae - bd} = 0.$$

Solving (8) and using $x = h$ produces (1).

Calculation of the extrema of the rational quadratic in Fig. 1.

Eq. (1) can be applied to calculate h and then the rational quadratic can be used to find $v = q(h) = \frac{h^2 + 6h + 3}{4h^2 + 5h + 6}$. However, there is no need to do so. Instead, the fact that $q(h) - \frac{h^2 + 6h + 3}{4h^2 + 5h + 6} = v - \frac{h^2 + 6h + 3}{4h^2 + 5h + 6} = 0$ has exactly two identical solutions is utilized. Hence,

$$v\left(4h^2+5h+6\right)-\left(h^2+6h+3\right)=0$$
$$\Rightarrow (4v-1)h^2+(5v-6)h+6v-3=0 \qquad (9)$$
$$\Rightarrow h^2+\frac{5v-6}{4v-1}h+\frac{6v-3}{4v-1}=0.$$

Applying the quadratic formula to (9) gives

$$h=-\frac{5v-6}{2(4v-1)}\pm\frac{1}{2}\sqrt{\left(\frac{5v-6}{4v-1}\right)^2-4\left(\frac{6v-3}{4v-1}\right)}. \qquad (10)$$

However, there are two identical solutions to (10). Hence,

$$\left(\frac{5v-6}{4v-1}\right)^2-4\left(\frac{6v-3}{4v-1}\right)=0 \qquad (11)$$

and

$$h=-\frac{5v-6}{2(4v-1)}. \qquad (12)$$

From (11),

$$(5v-6)^2-4(6v-3)(4v-1)=0$$
$$\Rightarrow 25v^2-60v+36-4\left(24v^2-6v-12v+3\right)=0$$
$$\Rightarrow -71v^2+12v+24=0 \qquad (13)$$
$$\Rightarrow v^2-\frac{12}{71}v-\frac{24}{71}=0.$$

Solving (13) gives the maximum at $v=0.672$ and the minimum at $v=-0.503$. Substituting these values into (12) gives the corresponding x coordinates, i.e. $h=0.782$ for the maximum and $h=-1.41$ for the minimum.

Conclusion

The conventional calculus formula for the x coordinates of the extrema of rational quadratics has been derived without using calculus. Furthermore, an alternative method of finding the extrema, by first finding the y coordinates and then the x coordinates has been presented.

Appendix

In this appendix, calculus is used to derive the formula for the x coordinates of the extrema points.
Recall,

$$q'(x)=\frac{\left(dx^2+ex+f\right)\left(2ax+b\right)-\left(ax^2+bx+c\right)\left(2dx+e\right)}{\left(dx^2+ex+f\right)^2}=0$$

$$\begin{aligned}
&\Rightarrow \left(dx^2+ex+f\right)\left(2ax+b\right)-\left(ax^2+bx+c\right)\left(2dx+e\right)=0\\
&\Rightarrow 2adx^3+bdx^2+2aex^2+bex+2afx+fb\\
&-\left(2adx^3+aex^2+2bdx^2+bex+2cdx+ce\right)=0 \qquad \text{(A1)}\\
&\Rightarrow \left(-bd+ae\right)x^2+\left(2af-2cd\right)x+fb-ce=0\\
&\Rightarrow x^2+\frac{2af-2cd}{-bd+ae}x+\frac{fb-ce}{-bd+ae}=0.
\end{aligned}$$

Therefore, using the quadratic formula,

$$x=\frac{-af+cd}{-bd+ae}\pm\frac{1}{2}\sqrt{\left(\frac{2af-2cd}{-bd+ae}\right)^2-4\left(\frac{fb-ce}{-bd+ae}\right)}. \qquad \text{(A2)}$$

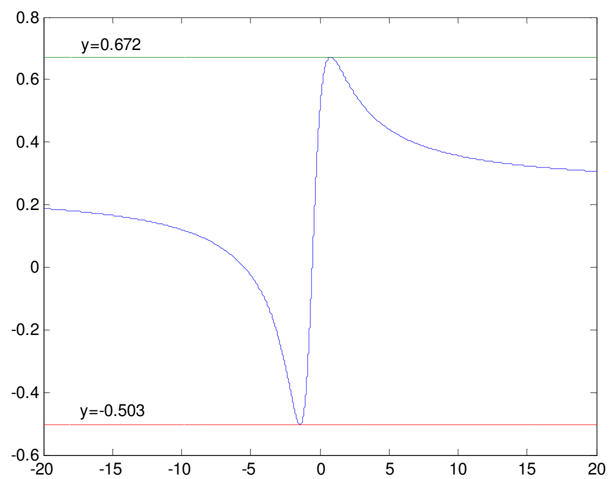

Fig. 1. Plot of the rational quadratic $q(x)=\frac{x^2+6x+3}{4x^2+5x+6}$. Note that the line of maximum $(y=0.672)$ and the line of minimum $(y=-0.503)$ each intercepts the

rational quadratic at exactly one point (more correctly, two identical points). All other lines of zero slope intercept the rational quadratic at two non-identical points.

† ***Kenneth V. Cartwright, Ph.D.***, College of the Bahamas, Nassau, Bahamas.

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DISCOVERING THE EXISTENCE OF FLAW IN THE PROCEDURE OF DRAWING ENLARGED EXPERIMENTAL CURVE

Pramode Ranjan Bhattacharjee †

Abstract

This paper identifies the flaw existing in the traditional procedure of drawing enlarged experimental curves. The usual practice of obtaining an enlarged curve by making use of proper scales with maximum enlargement along the two axes of coordinates has been found to be in conflict with the fundamental concept of "Enlargement" in Transformation geometry and is therefore flawed. With a view to getting rid of the aforesaid flaw and to establish a rationality between theory and practice, this paper emphasizes the need of using two similar proper scales with maximum enlargement along the two axes of coordinates for drawing enlarged experimental curves.

Key Words: Experimental graph; Transformation geometry; Enlargement.

INTRODUCTION

Experimental study of Science and Engineering is based on well-defined Laboratory guidelines/instructions. While working in a laboratory, students are always advised to follow the standard guidelines/instructions [3-12]. Now what about those guidelines/instructions which have no resemblance with the well-known theoretical concepts? It is a high time to think of such guidelines/instructions and to get rid of them with alternative flawless replacements, if possible or to get rid of them forever.

An examination of the traditional guidelines/instructions [3-12] in respect of drawing experimental graph has been made in this paper. In order to enhance graphical reliability or to minimize the value of one smallest division of the graph paper, the traditional concept [3-12] involved in experimental graph drawing is to make use of proper scale with maximum enlargement so as to cover up almost the entire portion of the graph paper used for drawing graph. Sometimes use of two different scales, each of which is a proper scale, is also insisted upon [4, 7, 12]. This very practice of artificial enlargement of a graph (figure) using dissimilar scales (each of which is a proper scale) in drawing experimental graph is flawed and it violates the fundamental concept of "Enlargement" in Transformation geometry [1, 2]. Such a curve is never an exact enlarged replica of the original curve (figure). In order to get rid of the aforesaid flaw and to establish a parity with the fundamental concept of "Enlargement" in Transformation Geometry, this paper emphasizes use of *two similar proper scales with maximum enlargement along the two axes of coordinates* in drawing experimental graphs.