On the Stability Properties of the Linear System of Ordinary Differential Equation

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Abstract

In this paper we study the stability of systems of the form

$$x^1 = F(t,x) \qquad (1.1)$$

Where $F: R_+ x R^2 \to R^n iscontinous.$

A solution $x_0(t), t\epsilon R_+$ of the system (1.1) is stableif the solutions of(1.1) which start close to $x_0(t)$ at the origin, remain close to $x_0(t)$, for all $t\epsilon R_+$ in a certain sense. This actually means that small disturbances in the system that effect small perturbations to the initial conditions of solutions close $x_0(0)$ do not really cause a considerable change to these solutions over the intervalR_+. The various concepts of stability that we study in this paper are actually dealing with the fashion in which the solutions close to $x_0(t)$initially behave on infinite subintervals of R.

Although there are numerous types of stability, we present here only five types that are most important in the applications of linear and perturbed linear systems.

Keywords: Differential system, solutions of systems of differential equations, stability of solutions, Eigen values of a system, Perturbed differential equation

1. DEFINITIONS OF STABILITY[1]: In the following definitions $x_0(t)$ will denote a fixed solution of (1.1) defined on [0, ∞).

Definition 1.1 The solution $x_0(t)$ is called "stable"for every$\varepsilon > 0$there exits$\delta(\varepsilon) > 0$ such that every solution,$x(t)$ of (1.1) with

$\|x(0) - x_0(0)\| < \delta(\varepsilon)$ exists and satisfied $\|x(t) - x_0(t)\| < \epsilon$ on R_+.The solution $x_0(t)$ is called "asymptotical", if it is stable and there exists constant $n > 0$ such that $\|x(t) - x_0(t)\| \to 0$ $as t \to \infty$ whenever $\|x_0(0) - x_0(0)\| n$.

The solution $x_0(t)$ is called "instable" if it is not stable.

Definition 1.2 The solution $x_0(t)$ is called "uniformly stable: if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that every solution, $x(t)$ of (1.1) with $\|x(0) - x_0(0)\| < \delta(\varepsilon)$ for some $t \geq 0$ exists and satisfied $\|x(t) - x_0(t)\| < \epsilon$ on $[t_0, \infty)$.It is called "uniformly asymptotically stable"if it ,is uniformly stable and there exists $n > 0$ with the property, for every $\varepsilon > 0$ there exists $T(\varepsilon) > 0$ such that $\|x(x_0) - x_0(t_0)\| < n$ for some $t_0 \geq 0$ implies $\|x(t) - x_0(t)\| < \varepsilon$ for every $t \geq t_0 + T(\varepsilon)$.

It is obvious that uniform stability implies stability and that uniform asymptotic stability implies asymptotic ability.

Definition 1.3 The solution $x_0(t)$ is called "strongly stable" if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that every solution $x(t)$ of (1.1) with $\|x(t_0) - x_0(t)\| < \delta(\varepsilon)$ for some $t_0 \geq 0$ exists and satisfies $\|x(t) - x_0(t)\| < \varepsilon$ on R_+.

Naturally, strong stability implies uniform stability. We should mention here that the definition of stability can be replaced by any but fixed interval $[t_0, \infty)$ of the real line. We should also mention that $x_0(t)$ can be considered to the zero solution. In fact, if $x(t) \equiv 0$ is not a solution of (1.1), then the transformation $u(t) = x(t) - y(t)$, where $y(t)$ is a fixed solution of (1.1) into the system.

$$u^1 = F\big(t, u + y(t)\big) - F\big(t, t(t)\big) = G(t, u) \qquad (1.2)$$

This system has the function $u(t) \equiv 0$ as a solution. The stability properties of this solution correspond to the stability properties of the solution. $u(t)$.

2. LINEAR SYSTEMS[2],[1]

In this section we study the stability properties of the linear systems.

$$\left.\begin{array}{r} x' = A(t)x. \\ x' = A(t)x + d(t) \end{array}\right\} \qquad (2.1)$$

where $A: R_+ \to M_n$, $f: \mathbb{R} \to \mathbb{R}^n$ arecontinuous. It is clear that the solution $x_0(t)$ of the differential function of (2.1)satisfiesone of definitions of stability of the previous section if and only if the zero solution of (2.1) has the same property. This follows from the fact that stability involves differences of solutions, combined with the superposition principle. Consequently, we may talk about the stability of the differential function of (2.1) instead of the stability of one of its particular solutions. This will be done in the sequel even if $f = 0$.

Theorem 2.1 Let $X(t)$ be a fundamental matrix of (2.1). Then (2.1)is stable if and only if there exists a constant $K > 0$ with.

$$\|X(t)\| \le K, t \in R_+ \qquad (2.2)$$

The system (2.1) is asymptotically stable if and only if

$$\|X(t)\| \to 0 \; as t \to \infty$$

The system (2.1) is uniformly stable if and only if there exists a constant $K > 0$ with.

$$\|X(t)X^{-1}(s)\| \le K, 0 \le s \le t < +\infty \qquad (2.3)$$

The system (2.1) is uniformly asymptotically stable if and only if there exist a constant $K > 0$ such that.

$$\|X(t)X^{-1}(s)\| \le Ke^{-a(t-s)}, 0 \le s \le t < +\infty \qquad (2.4)$$

The system (2.1) is strongly stable if and only if there exists a constant K>0 such that:

$\|X(t)\| \leq K, \|X^{-1}(t)\| \leq K, t \in R_+$ (2.5)

Proof: We may assume that $(X(0)) = 1$ because the conditions in the hypothesis hold for any fundamental matrix of (2.1) ,if they hold for a particular one. Assume first that (2.2) holds, and let $x(t) \in R_+$, be a solution of (1.1) with $x(0) = x_{0+}$. Then, since $x(t) = X(t)x_0$ if for a given $\varepsilon > 0$ we choose $\delta(\varepsilon) = K$, we have.

$\|x(t)\| = \|X(t)x_0\| < \varepsilon \, for \, \|x_0\| < \delta(\varepsilon)$

Thus, System (2.1) is stable conversely, suppose hat (2.1) is stable and Fix $\varepsilon > 0, \delta(\varepsilon)\, 0$ with the property.

$\|X(t)x_0\| < \varepsilon$

For every $x_0 \in R^n$ with $\|x_0\| < \delta(\varepsilon)$.For a fixed $t \in R$, we get

$[1/\delta(\varepsilon)]\|X(t)x_o\| = \frac{X(t)x_o}{\delta(\varepsilon)} < \in \quad /\delta(\varepsilon)$ (2.6)

Since $x_0\delta(\varepsilon)$ ranges over the interior of the unit ball, we obtain

$\|X(t)\| = \sup_{\|u\|_{<1}} \|X(t)\,u\| < \delta(\varepsilon)$ (2.7)

This completes the proof of the first case because (2.7) holds for arbitrary $t \in \mathbb{R}$.

Now assures that (2.7) holds. Then (2.2) holds for some K>0 and

$\lim_{n\to\infty} x(t) = \lim_{n\to\infty} X(t)\, x_0 = 0$

For any solution $x(t)$ of (2.1)with $x(0) = x_0$. Thus (2.1) is asymptotically stable. Conversely, assume that (2.1) is asymptotically stable. Then there exists $n > 0$ such that $X(t)x_0 \to 0$ for every $x_0 \epsilon R^n$ with $[\![x_0]\!] \leq n$. Choose

$$x_0 = \begin{bmatrix} n \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

Since $X(t)\,x_0 = ny(t)$, where $y(t)$ is the first column $X(t)$, we obtain that every entry of first column of $X(t)$ tends as $t \longrightarrow \infty$. similarly one conclude that every entry of $X(t)$ tends to zero as $t \longrightarrow \infty$. This completes the proof of this case. In order to prove the third conclusion of the theorem, let (2.1) hold and let $t_0 \epsilon R_+$ be given. Then $x(t) = X(t)X^{-1}(t_0)x_0$ is the solution of (2.1) with $x(t_0) = x_0$. Thus,

$\|x(t)\| \leq \|X(t)X^{-1}(t_0)\|\|x_0\| \leq K\|x_0\|$

for any $x_0 \epsilon R^n$ with $\|t_0\| < K^{-1}\varepsilon$, proves the uniform stability of (1.1) with $\delta(\varepsilon) = K^{-1}\varepsilon$.Now assume that (1.1) is uniformly stable. Fix $\in > 0, \delta(\varepsilon) > 0$ such that $\|X(t)R^{-1}(t_0)x_0\| < \epsilon$ for any $x_0 \in R^n$ with $\|x_0\| < \delta(\varepsilon)$, any $t_0 \in R_+$ and any $t \geq t_0$.from this point on, the proof follows as the sufficiency part of the first case and is therefore omitted.

In the fourth case let (2.3) hold. Then (2.1) is uniformly stable by virtue of (2.3). Now let $\varepsilon(0 < \varepsilon < K), t_0 \in R_+$ be given, and let $x(t)$ be the solution of (2.1) with $\|x(t_0)\| = \|t_0\| < 1$.

$\|x(t)\| = \left\|X(t)R^{-1}(\frac{t_0}{x_0})\right\| < K ex^{-f} - \alpha(t - t_0)] \leq \varepsilon$

for every $t \geq t_0 + T(\varepsilon)$,where $T(\varepsilon) = \alpha^{-1}I_n(\varepsilon/K)$. Consequently, System (2.1) uniformly asymptotically stable. Fix $\varepsilon(0 < \varepsilon < u), T = T(\varepsilon), n$ as in definition (2.6). Then $\|x_0\| = n$ implies.

$\|X(t)X^{-1}(t_0)t_0\| < \varepsilon, t \geq t_0 - T$

Consequent working as in the firstcase, we find

$\|X(t)X^{-1}(t_0)\| \leq \mu < 1, t \geq t_0 + T$

Where μ ,= ε/n. Now (2.3) implies the existence of a constant $K > 0$ such that

$\|X(t)X^{-1}(t_0)\| \leq K, t \geq t_0$

Thus, given $t \geq t_0$, there exists an integer $m \geq 0$ such that $t_0 + mT \leq t \leq t_0 + (m+1)T$ it follows that

$$X(t)X^{-1}(t_0) = X(t)X^{-1}(t_0 + mT)X(t)X^{-1}(t_0 + mT)X(t)X^{-1}(t_0 + (m-1)T)$$

$$X(T_0 - (M-1)T) \ldots X^{-1}(t_0 + T)X(t_0 + T)X^{-2}(t_0)\|$$

$$\leq \|X(t)X^{-1}(t_0 + mT)\|\|X(t_0 + mT)X^{-1}(t_0)\|$$

$$\ldots \|X(t_0 + mT)X^{-2}(t_0)\|$$

$$\leq K\mu^m$$

If we take $\alpha = -T^{-2}In\mu$, then

$$\|X(t)X^{-1}(t)\| \leq \mu^{-1}K\mu^{m+1} = \mu^{-1}K\mu^{-(m+1)\alpha T}$$

$$\leq \mu^{-1}K\varepsilon^{-\alpha(t-t_0)}$$

For every $t \geq t_0 + T$. Thiscompletes theproof of the case of uniform asymptotic stability.

Assume now that (2.5) holds and, given $\varepsilon > 0$,choose $\delta(\varepsilon) = K^{-2}\varepsilon$, then we have

$$\|X(t)X^{-2}(t_0)x_0\| \leq \|X(t)X^{-2}(t_0)\| \; x_0\|$$

$\leq K\|t_0\| < \varepsilon$

Whenever $\|x_0\| < K^{-2}\varepsilon$ and $t_1 t_0 \in R_+$. Thus system (2.1) is strongly stable. To show the converse, let (1.1) be strongly stable and fix $\varepsilon > 0, \delta(\varepsilon) > 0$ such that

$$\|X(t)X^{-1}(t_0)x_0\| < \varepsilon, t, t_0 \in R_+ \quad (2.8)$$

Whenever $\|x_0\| < \delta(\epsilon)$. In fact, this follows from (2.8) if we take $\|X(t)X^{-1}(t_0)x_0\| < \varepsilon$

Provided that $\|x_0\| < \delta(\epsilon)$. In fact, this follows from (2.8) if we take $t_0 = 0, t = 0$ respectively. Thus, as above,

$\|X(t)\| \leq \varepsilon/\delta(\varepsilon), \|X^{-1}(t)\| \leq \varepsilon/\delta(\varepsilon)$

It follows that (2.5) holds for $K = \varepsilon/\delta(\varepsilon)$.

Before we consider System (2.1)with a constant matrix A, we should note that in the caseof an "autonomous" system (that is, $F(t,x) = F(x)$) stability is equivalent to uniform stability and asymptotic stability is equivalent to uniform asymptotic stability. This is consequence of the fact that in this case $y(t) = x(t+\alpha)$is a solution of (1.1) if$x(t)$is a solution. This is true for any number $\alpha \in R_+$. Now consider the system

$$x' = Ax \quad (2.9)$$

With $A \in M_n$, If λ is an eigenvalue of A, then the dimension of the "Eigenspace" ofλ (the subspace of C" generated by the eigenvectors of A corresponding toλ) is called "index" of λ. The following theorem characterized the fundamental matrices of (2.9).

Theorem 2.2: Let $X(t) = \varepsilon^{tA}$ be the fundamental matrix of (2.9) with $X(0) - I$. Then every entry $X(t)$ takes the form $\varepsilon^{\alpha t}[p(t)\cos\beta t - q(t)\sin\beta t]$ or the form $\varepsilon^{\alpha t}p(t)\sin\beta t + q(t)\cos\beta t)$, where $\lambda = \alpha + Bi$ is some eigenvalue of A and p,q are real polynomials in t. the degree d of the polynomial $p(t) + iq(t)$ satisfies $0 \leq d \leq m - r$, where m is the multiplicity of λ and r its ,index. Furthermore, if $m \neq r$, there is at least one entry of $X(t)$ such that $d \neq 0$.

Now we are ready to establish the stability properties of (2.9) in terms of the Eigenvalue of the matrix A.

Theorem 2w.3: The system (2.9) is stable and if and only if every Eigenvalue of A that has multiplicity m equal to its index r has nonpositive real part, and every other Eigenvalue has negative real part. The system (2.9) is asymptotically stable if and only if every Eigenvalue of A has negative real part. It is strongly stable if and only if Eigenvalue of A is purely imaginary and multiplicity equal to its index.

Proof. Let $X(t) = \varepsilon^{tA}, t \in R_+$ Then (2.9) is stable if and only if $\|X(t)\| \leq K$. where $K > 0$ is a constant. Now let $\lambda = \alpha + Bi$ be an Eigenvalue of A, then every entry of $X(t)$ corresponding to λ will be bounded if and only if $\alpha \leq 0$ for $m = r$ and $\alpha < 0$ for $m > r$. This completes the proof of our first assertion. The system (2.9) is asymptotically stable if and only if $\varepsilon^{tA} \to 0$ as $t \to \infty$. This is of course possible if and only if every Eigenvalue of A has negative real part. The system is strongly stable if and only if there exists a constant $K > 0$ such that $\|\varepsilon^{tA}\| \leq K, \|\varepsilon^{-tA}\| \leq K$ for every $t \in R_+$.Since ε^{-tA} solves the system $X' = -AX$, and λ is an Eigenvalue of A if and only if $-\lambda$ is an

Eigenvalue of $-A$, these inequalities can hold if and only if every Eigenvalue of A has zero real part and $m = r$.

3. THE MEASURE OF A MATRIX; FURTHER STABILITY CRITERIA[1],[3]

Definition 3.1: Let A be an $n \times n$ matrix. Then $\mu(A)$ Denotes the "measure of A" which is defined by

$$\mu(A) = \lim_{h \to \infty} \frac{\|I + hA\| - 1}{h}$$

Theorem 3.1: The measure $\mu(A)$ exists as a finite number for every $A \in M_n$.

Proof. Let ε be given such that $0 < \varepsilon < 1$, and consider the function

$$g(h) = \frac{\|I + hA\| - 1}{h} \quad h > 0$$

Then we have

$$\|I + \varepsilon hA\| = \|\varepsilon(I + \varepsilon hA) + (1 - \varepsilon)I\| \le \varepsilon\|I + hA\| + (1 - \varepsilon)$$

or

$$g(h) = \frac{\|I + \varepsilon hA\| - 1}{\varepsilon h} \le \frac{\|I + hA\| - 1}{h} = g(h)$$

Thus, $g(h)$ is an increasing function of h. On the other hand

$$\frac{\|I + hA\| - 1}{h} = \frac{\|I + hA\| - 1}{h} \le \frac{I + hA - I}{h} = g(h)$$

This impliesthe existence of the limit of the function $g(h)$ as $h \to 0^+$, it follows that $\mu(A)$ exists and is finite.

Theorem 3.2: Let $A \in M_n$ be given. Then $\mu(A)$ hasthe following properties:

(i.) $\mu(\alpha A) = \alpha\mu(A)$ for any $\alpha \in R_+$

(ii.) $\|\mu(A)\| \le \|A\|$

(iii.) $\mu(A+B) \leq \mu(A) + \mu(B)$

(iv.) $\|\mu(A) - \mu(B) \leq \|A - B\|$

Proof. Case (i) is trivial and (ii)follows from the fact, that $\|g(h)\| \leq \|A\|$for all $h > 0$, where$g$ is as in the proof of Theorem (3.2). Inequality (iii) follows from

$$\frac{\|I + h(A+B)\| - 1}{h} \leq \frac{\|(1/2)I + hA\| - (1/2)}{h} + \frac{\|(1/2)I + hB\| - (1/2)}{h}$$

$$= \frac{\|I + 2hA\| - 1}{2h} + \frac{\|I + 2hB\| -}{2h}$$

Inequality (iv) follows easily from (ii) and (iii).

The following theorem establishesthe relationship between the solution of (2.1) and the measure of the matrix $A(t)$.

Theorem 3.3: Let $A: R_+ \to M_+$ becontinuous. Then for every $t_0,\ t \in R_+$ with $t \geq t_0$ we have

$$\|x(t_0)\| \exp[-\int_{t_0}^{t} \mu(-A[s])ds] \leq \|x(t)\| \leq \|x(t_0)\| \, exp \int_{t_0}^{t} \mu(A(s))ds]$$

Where $x(t)$is any solution of (2.1).

Before we provide a proof of Theorem 3.3, we establish the auxiliary lemma

Lemma 3.1: Let $r:(t_0, b) \to R_+,\ \emptyset: [t_0, b) \to R\ (0 \leq t_0 < b \leq +\infty)$ be continuous and such that

$r:(t) \leq \emptyset(t) r(t),\ \ t \in [t_0, b)$

Where $r:$ denotes the rightderivative of the function $r(t) \leq u(t), t \in [t_0, b)$, where$u(t)$ is the solution of

$$u' = \emptyset(t)u, u(t_0) = r(t_0) \qquad (3.1)$$

Proof: Let$f_1 \in (t_0, b)$ be an arbitrary point. We will show that $r(t) \leq u(t)$on the interval $[t_0, t_1]$.Consider first the solution $u_{,}(t), t \in [t_0, t_1]$ of the problem

$$u' = \emptyset(t)u + {}^{1}/_{n},\ u(t_0) = r(t_0), n = 1, 2, \ldots \qquad (3.2)$$

respectively, Fix n and assume the existence of a point$t_2 \in (t_0, t_1)$ such that $t(t_2) > u_n(t_2)$. then there exists$t_2 \in (t_0, t_1)$ such that $r(t_0) = u_0(t_0)$ and$r(t) > u_0(t_0)$ on $(t_3, t_2]$. From (3.2), we obtain that

$$n_n^1(t_3) = \emptyset(t_3)u_n(t_3) + {}^{1}/_{n}$$
$$= \emptyset(t_3)r(t_3) + {}^{1}/_{n}$$
$$\geq r_+^1(t_3) + {}^{1}/_{n}$$
$$> r_+^1(t_3)$$

Consequently, $u_n(t) > r(t)$ is small right n eighborhood of the point t_3. This is a contradiction to $r(t) > u_n(t)$ on$(t_3, t_2]$. Thus$r(t) \leq u_n(t)$ for any $t \in (t_0, t_1]$and any $n = 1, 2, \ . \ . \ .$ Now we use Gronwall's inequality to show that (3.2), actually implies

$$\|u_n(t) - u_m(t)| \leq 2t_1| {}^{1}/_{n} - {}^{1}/_{m} |\exp\{\int_{t_0}^{t} |\emptyset(s)|ds\|, t \in [t_0, t_1]$$

For $n, m \geq 1$. thus, the sequences $\{u_p(t)\}, n = 1, 2, \ldots$ is Cauchy. It follows that $u_n(t) \to u(t)\ as\ n \to \infty$ uniformly on $[t_0, t_1]$ where $u(t)$is the solution problem (3.1)on the interval $[t_0, b)$. Since t_1 is arbitrary, it follows that $r(t) \leq u(t), t \in [t_0, b)$.

It should be noted that a corresponding inequality holds if $r_-^1(t)$ is the left derivative of $t(t)$ on $[t_0, b)$.

Proof ofTheorem 3.3: $r(t) = \|x(t)\|$. We are planning to show that

$$r(t) \le \mu(A(t))r(t)$$

To this end, we first notice that for any two vectors $x_1, x_2 \in R^+$ that limit

$$\lim_{h \to 0^+} \frac{\|x_1 + hx_2\| - \|x_1\|}{h}$$

exists as a finite number. To show this, it suffices to show that the function

$$g_1(h) \frac{\|x_1 + hx_2\| - \|x_1\|}{h}$$

is increasing and bounded by $\|x_2\|$ on $(0, \infty)$. We omit the proof, which is very similar to the corresponding one for the function $g(h)$ in the proof of Theorem 3.1. It follows that the limit.

$$\lim_{h \to 0^+} \frac{\|x(t) + hx'(t)\| - \|x(t)\|}{h} \qquad (3.3)$$

exists as a finite number. We will show that this number equals $r^{+1}(t)$. In fact, let $h > 0$ be given. Then we have

$$\left|\frac{\|x(t+h)\| - \|x(t)\|}{h} - \frac{\|x(t) + hx'(t)\| - \|x(t)\|}{h}\right|$$

$$= \left|\frac{\|x(t+h)\| - \|x(t) + hx^1(t)\|}{h}\right|$$

$$= \frac{\|x(t+h) - x(t) - hx^1(t)\|}{h} \to 0 \text{ as } h \to 0^+$$

which proves (3.3) and consequently

$$r^{+1}(t) = \lim_{h \to 0^+} \frac{\|x(t) + hA(t)x(t)\| - \|x(t)\|}{h}$$

$$\le \lim_{h \to 0^+} \frac{\|I + hA(t)\| - 1}{h}$$

$$\le \mu(A(t))r(t)$$

Applying Lemma 4.11, we obtain

$$\|x(t)\| = r(t) \leq \|x(t_0)\| \exp\left\{\int_{t_0}^{t} \mu\left(A(s)\right) ds\right] \qquad (3.5)$$

for every $t \geq t_0$. In order to find corresponding lower bound of $\|x(t)\|$, let $u = -t, u_0 = -t_0$. then $y(u) = x(-u), u \in (-\infty, u_0]$, satisfies the system.

$$y^1 = -A(-u)y$$

Thus, as in (3.5), we get

$$\|y(u_0)\| \leq \|y(u)\| \exp\left\|\int_{u}^{u_0} \mu(-A(-s)\, ds\right\|$$

$$= \|y(u_0)\| \exp\{-\int_{-u}^{u_0} \mu(-A(v))s)\, dv\}, u_0 \geq u$$

or

$$\|x(t_0) \leq \|x(t)\|\| \exp\left[-\int_{-u}^{u_0} \mu(-A(v)s)\right], u_0 \geq u$$

$$\leq \|y(x+h)\| - x(t) - hx'(t) \to 0 \; as \; h \to 0^+$$

This completes the proof,

We are now ready for the main theorem of this section

Theorem 3.4: Consider system (2.1) with $A: R_+ \to M_+$ continuous

If

$$\lim_{t\to\infty} \inf \int_0^t \mu\left(-A(s)\right) ds = -\infty$$

then (1.3) is unstable if

$$\lim_{t\to\infty} \sup \int_0^t \mu\left(-A(s)\right) ds < +\infty$$

then (1.3) is stable if

$$\lim_{t\to\infty} \int_0^t \mu\left(-A(s)\right) ds = -\infty$$

then (1.3) is asymptotically stable if

$\mu(A(t))ds = -\infty$

then (1.3) is uniformly stable if for some $r > 0$,

$\mu(A(t)) \leq -r, t \geq 0$

then (1.3) is uniformly asymptotically stable.

Table 4.13

$\|x\|$	$\mu(A)$
$\|x\|_1$	
Largest Eigenvalueof $1/2\,(A + A^T)$	
$\|x\|_2$	Max $\{a_{ii} + \sum_{f,\, j \neq i} a_{ij}\}$
$\|x\|_3$	Max $\{a_{ff} + \sum_{f,\, j \neq i} a_{ij}\}$

4. PERTURBED LINEAR SYSTEMS[4],[7],[1]

In this section we study the stability of systems of form

$$x' = (A(t)x + F(t,x) \qquad (4.1)$$

where $A: R_+ \to M_n$, $F: R^n \to R^n$ are continuous functions with $F(t,0) \equiv 0, t \in R_+$. We start with a theorem concerning the asymptotic stability of(S_F) the differential function of (1.1). The proof of this theorem is based on Lemma (3.1).

Lemma 4.1: Let $X(t)$ be a fundamental matrix of the system (2.1). Assume further that there exists a constant $K > 0$ such that

$$\int_0^t \|X(t)X^1(s)\|ds \leq K, t > 0 \qquad (4.2)$$

Then there exists a constant $M > 0$ such that

$$\|X(t)\| \leq Me^{(-k-1)t}, t \geq 0$$

Proof. Let $u(t) = \|X(t)\|^{-1}$. Then we have

$$\int_0^t u(s)\,ds)X(t) = \int_0^t u(s)ds)X(t)X^{-1}(s)X(s)$$

from which we obtain

$$\int_0^t (u(s)ds)\|X(t)\| \leq \int_0^t \|X(t)X^{-1}\|\|X(s)\|y(s)\,ds \leq K, t \leq 0$$

$$u(t) \geq K^{-1}\int_0^t u(s)\,ds) \qquad (4.3)$$

Now let $A(t)$ denote the integral on the right hand side of (4.3). Then we have

$$\lambda'(t) \geq K^{-1}\lambda(t), t \geq 0 \qquad (4.4)$$

Dividing (4.4) by $\lambda(t)$ and integrating from t_0 $to > t_0$, we obtain

$$\lambda(t) \geq \lambda(t_0)e^{k^{-1}}t^{-t_0)}, t \geq t_0$$

Consequently,

$$\|X(t)\| = u(t)]^{-1} \leq K[\lambda(t)]^{-1} \leq [K/\lambda(t_0)e^{k^{-1}}(t^{-1_0})$$

for every $t \geq t_0$. We choose M so large that both

$$M \geq [K/\lambda(t_0)]e^{k^{-1t_0}}$$

and

$$\|X(t)\| \leq Me^{k^{-1t_0}}, 0 \leq t \leq t_0$$

This completes the proof

Theorem 4.2: Let $X(t)$ be a fundamental matrix of (2.1) such that

$$\int_0^t \|X(t)X^{-1}(s)\|ds \leq K, t \geq 0$$

Moreover, let

$$\|F(t,x)\| \leq \mu\|x\| \ t \geq 0$$

with μ stratifying $0 \leq \mu < K^{-1}$. Then the zero solution of (S_F) is asymptotically stable.

Proof: Let $X(x)$be the fundamental matrix of (S) with$X(0) = I$.then since $X(x)X^{-1}(s) = Y(t)Y^{-1}(s)$ forany otherfundamental matrix$Y(t)$of(S). Lemma 4.1 holds for this particular$X(t) \to 0 \ as \ t \to \infty$. If $x(t)$ is a local solution of the differential function of (2.1) defined to right of $t = 0$, then $x(t)$satisfies the system

$$u^t = A(t)u + F(t, x(t))$$

Using the variation of constants formula for the system we obtain

$$x(t) = X(t)x(0) + \int_0^t X(t)X^{-1}(s)F(s, X(s))ds \quad (4.5)$$

Letting $L > 0$be such that $\|X(t)\| \leq L \ for \ t \geq 0$,we obtain

$$\|x(t)\| \leq L\|x(0)\| + \mu K \max_{0\leq + \leq t} x(s)$$

which implies

$$\max_{0\leq + \leq t} \|x(s)\| \leq (1 - \mu K)^{-1} L \, x(0)$$

It follows that

$$\|x(t)\| \leq (1 - \mu K)^{-1}$$

as long as $x(t)$ defined. This implies that $x(t)$ is continuable $to +\infty$ and the zero solution is stable. Now we show that $x(t) \to 0$ as $t \to \infty$. To this end, let

$$c = \lim_{t\to\infty} sup \, \|x(t)\|$$

and pick d such that $\mu K < d < 1$. If $c > 0$, the, since $d\ c > c$, there exists $t \geq 0$ such that

$$\|x(t)\| \leq d^{-1}c$$

for every $t \geq t_0$, Thus, (4.5) implies

$$\|x(t)\| \quad = X(x)\,x(0) + X(t)\int_0^t X^{-1}(s)F(s, x(s))ds$$

$$= \left\| x(t)\,x(0) + X(t)\int_0^t X^{-1}(s)F(s, x(s))ds + X(t)\int_0^t X^{-1}(s)F(s, x(s)) \right\| ds$$

$$\leq \|X(t)\|\|x(0)\| + \|X(t)\| \int_0^t \|X^{-1}(s)F(s, x(s))\| ds$$

$$+ \int_0^t \|X(t)X^{-1}(s)\| \, \|F(s, x(s))\| ds$$

$$\leq \|X(t)\|\|x(0)\| + \|X(t)\| \int_0^t \|X^{-1}(s)F(s, x(s))\| ds + \mu K d^{-1}c$$

Taking the lim sup above as $t \to \infty$, we obtain $c \leq \mu K d^{-1}c$; that is, a contradiction. Thus, $c = 0$. This completes the proof.

The following theorem has a corollary concerning the uniform stability of system solution of the differential function of (2.1).

Theorem 4.3 $\|X(t)X^{-1}(s)\| \leq K, t \geq s \geq 0$

where K is positive constant. Moreover, let

$$F(t, x) \leq \lambda(t)\|x\|$$

where $\lambda : R_+ \to R_+$ is continuous and such that

$$\int_0^{\infty} \lambda/(t)\,dt < \infty$$

Then if

$$M = K\,exp\,(K\int_0^{\infty} \lambda/(t)\,dt$$

every local solution $x(t)$ ofsolution of the differential function of (2.1), defined to the right ,of the point $t_0 \geq 0$, is continuable $t_0 + \infty$ and stratifies

$$\|x(t)\| \leq M\|x(t_0)\|$$

for every $t \geq t_0$.

Proof: From the variable of constants formula (4.5), with t_0 replacing 0, we have

$$\|x(t)\| \leq K\|x(t_0)\| + K\int_0^{\infty} \lambda/(t)\,dt$$

for $t \geq t_0$. Applying Crownwall's inequality, we obtain

$$\|x(t)\| \leq Kx\left(\|x(t_0)\|\exp\ (K\int_{t_0}^{\infty} \frac{\lambda}{(t)\,ds}\right) \leq M\|x(t_0)\|, for\ t \geq t_0 \quad (4.6)$$

Corollary 4.4: If the system (1.1) is uniformly stable, and if F is as in Theorem 2.2, then the zero solution of the differential function of (1.1)is uniformly stable. In particular, the uniform, stability of (2.1)implies the uniform stability of the system.

$$x^1 = [A(t) + B(t)]x$$

where $B: R_+ \to M_n$ iscontinuous and such that

$$\int_0^{\infty} \|B(t)\|ds < +\infty$$

The uniform asymptotic stability of the system the differential function for (1.1) follows theorem 4.5.

Theorem 4.5: Let $X(t)$ be a fundamental matrix of (2.1) such that

$$\|X(t)X^{-1}(s)\| \leq Ke^{-\mu(t-s)}\ t \geq s \geq 0$$

where$K_t\ \mu$ are positive constants. Let

$$\|F(t,x)\| \leq \lambda\ \|x\|$$

with λ, a positive constant, satisfying $\lambda < X^{-1}\mu$. Then if $c = \mu - \lambda K$, every solution $x(x)$ ofthe differential function for (1.1), defined in a right neighborhood oft_0 exist for $t \geq t_0$ and satisfies

$$\|X(t)\| \leq Ke^{-\varepsilon\|t;s\|}\|x(s)\|$$

for every t, s with$t \geq s \geq t_0$

Proof: From the variation of constants formula.

$$x(t) = X(t)X^{-1}(t_0) + \int_0^{\infty} X(t)\,K^{-1}(S)F\big(s, x(s)\big)ds$$

in a right neighborhood of the point$t_0 \geq 0$, we obtain

$$\|x(t)\| \leq Ke^{-\mu(t-t_0)}\|x(t_0)\| + \lambda\ K\int_{t_0}^{t} e^{-\mu(t-t_0)}\|x(s)\|ds\ t \geq t_0$$

Consequently, if $z(t) = e^{-\mu(t-t_0)}\|x(t)\|$, we obtain

$$z(t) \leq Kz(t_0) \lambda K\int_{t_0}^{t} z(s)\,ds, t \geq t_0$$

An application of Corwnwall's inequalityyields

$$z(t) \leq Kz(t_0)e^{-\lambda K(t-t_0)}$$

for $t \geq t_0$, and

$$\|x(t)\| \leq K\|x(t_0)\|e^{-c(t-t_0)}$$

Obviously, $x(t)$ is continuable $t_0 + \infty$ (see Theorem 3.8)

Corollary 4.6: If (2.1) is uniformly asymptotically stable and if the constant λ of Theorem 4.5 is sufficiently small, then the zero solution of the differential function of (1.1) is uniformly asymptotically stable the differential function of (1.3). In particular, the uniform asymptotic stability of the system (2.1) implies the same property for the system (4.6) where $B: R_+ \rightarrow M_n$ is continuous and such that $B(t) \rightarrow 0$ as $t \rightarrow \infty$.

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