Intersections of Tripods in Homogeneous Metric Continua

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Abstract

A brief history of results related to the contents of the paper is reviewed. Relevant definitions and notations are provided due to significant variation in the literature. Preliminary results are developed for a special class of homogeneous continua. A relation is then defined on an arbitrary collection of subsets of a topological space. It is verified that this relation is a partial order. Finally, it is shown that each point in a specific type of homogeneous metric continuum is the intersection of a maximal chain from a particular class of subcontinua relative to the partial order established above.

Introduction

Following the work of F. Burton Jones [6], Forest W. Simmons showed in October 1980 that if a homogeneous continuum is separated by some pair of points, then it is separated by each pair of its points, and thus is a Hausdorff circle [9, Main Theorem]. Richard A. Winton generalized several of Simmon's preliminary results ([12, Lemma 2],[12, Corollary 5],[12, Theorem 6]) in September 2007. In February 2009 Winton established the precise conditions under which subcontinua with finite boundaries in a homogeneous continuum can be separated by a single point ([13, Theorem 5],[13, Theorem 6]). Finally, in September 2010 Winton showed that in a certain class of homogeneous metric continua, the boundaries of a specific category of subcontinua form a partition of the collection of all boundary points of such subcontinua [14, Theorem 5].

We now proceed with definitions which are fundamental for the results that follow. Notations for these concepts are also established since they are not uniform throughout the literature. Included in these definitions and notations are that of a continuum; a metric continuum; the interior, boundary, and closure of a subset of a topological space; an n-pod of a space, where n is an integer greater than 1; the separation of a subset of a space; a space being separated by a subset; and the separation and pod numbers of a space.

Basic Definitions

In the most general sense, a continuum is a compact, connected, Hausdorff topological space. As a special case, a metric continuum is a compact, connected metrizable space.

If H is a subset of a topological space X, then Int(H), Bd(H), and Cl(H) are the topological interior, boundary, and closure of H in X, respectively. A separation $A|B$ of H is a partition of H into nonempty relatively open sets A and B. Furthermore, H separates X if and only if X is connected but X−H is not connected. If n is a positive integer, then n is the separation number of X, denoted by S(X), if and only if X contains a subset with n points which separates X, but X contains no subset with less than n points which separates X. In other words, S(X) is the minimal number of points required to separate X.

If n is an integer and $n > 1$, then H is an n-pod of X if and only if H is a subcontinuum of X whose boundary contains precisely n points. In particular, a 2-pod in X will be called a bipod, while a 3-pod in X will be referred to as a tripod. An abipodic space is a topological space which contains no bipods. Finally, n is the pod number of X, denoted by P(X), if and only if X contains an n-pod but X contains no k-pod whenever k is an integer and $1 < k < n$.

Preliminary Results

Subcontinua with finite boundaries in topological spaces have some very interesting properties. For example, distinct subsets, or for that matter even distinct subcontinua, in a topological space may have the same interior. However, when two n-pods in a Hausdorff space have a common interior, then they also have a common boundary, and hence must be identical.

Theorem 1: Suppose X is a Hausdorff space, n is an integer, $n > 1$, H and K are n-pods in X, and Int(H) = Int(K).
(a) Then Bd(H) = Bd(K).
(b) Consequently H = K.

Proof:

(a) Since H is an n-pod in X then Bd(H) = $\{p_i\}_{i=1}^{n}$ for some $\{p_i\}_{i=1}^{n} \subseteq X$. Since H and K are compact subsets of the Hausdorff space X then H and K are closed in X ([1, p. 81, Corollary 5.13],[3, p. 165, Theorem 6.4]). Therefore H = Cl(H) ([8, p. 83, Theorem 4.6],[11, p. 25, Theorem 3.7(e)]) = $\text{Int}(H) \cup \text{Bd}(H)$ [4, p. 72, Theorem 4.11(3)], while $\text{Int}(H) \cap \text{Bd}(H) = \varnothing$ since $\{\text{Int}(H), \text{Bd}(H), X-H\}$ is a partition of X ([2, p. 142, Theorem 30.2],[4, p. 72, Theorem 4.11(4)]). By a similar argument $K = \text{Cl}(K) = \text{Int}(K) \cup \text{Bd}(K)$ and $\text{Int}(K) \cap \text{Bd}(K) = \varnothing$ as well.

Suppose $1 \le i \le n$. Since H is connected then H has no isolated points. Therefore p_i is a cluster point of Int(H) = Int(K), and so $p_i \in \text{Cl}(K)$ ([4, p. 71, Theorem 4.7],[11, p. 35, Theorem 4.10]) = $\text{Int}(K) \cup \text{Bd}(K)$. However, since $p_i \in \text{Bd}(H)$ and $\text{Int}(H) \cap \text{Bd}(H) = \varnothing$, then $p_i \notin \text{Int}(H) = \text{Int}(K)$, so that $p_i \in \text{Bd}(K)$. Thus $p_i \in \text{Bd}(K)$ for $1 \le i \le n$, and so Bd(K) = $\{p_i\}_{i=1}^{n}$ = Bd(H) since K is an n-pod in X.

(b) Since Int(H) = Int(K) by the hypothesis, and Bd(H) = Bd(K) by part (a), then $H = \text{Int}(H) \cup \text{Bd}(H) = \text{Int}(K) \cup \text{Bd}(K) = K$.

The next few results deal with abipodic spaces which contain tripods. We now show that an abipodic space contains a tripod if and only if its separation number is precisely 3. In this case the pod number of the space is also 3.

Lemma 2: Suppose X is a homogeneous continuum.
(a) Then $S(X) \geq 2$.
(b) X is abipodic if and only if $S(X) \geq 3$.
(c) X is abipodic and contains a tripod if and only if $P(X) = S(X) = 3$.

Proof:
(a) Since X is a homogeneous continuum, then no single point can separate X [13, Theorem 2]. Therefore $S(X) \geq 2$.

(b) Suppose X is abipodic. If $S(X) = 2$, then there exist $x,y \in X$ such that $\{x,y\}$ separates X. Consequently, there exist $A \subseteq X$ and $B \subseteq X$ such that $A|B$ is a separation of $X-\{x,y\}$. Therefore $A\cup\{x,y\}$ and $B\cup\{x,y\}$ are bipods in X with common boundary $\{x,y\}$ ([9, Lemma 1],[12, Lemma 2]). This is a contradiction since X is abipodic, and so $S(X) \neq 2$. Since $S(X) \geq 2$ by part (a), then $S(X) \geq 3$.

Conversely, suppose that $S(X) \geq 3$. If X contains a bipod H, then Bd(H) separates X [12, Lemma 3]. Therefore $S(X) \leq |Bd(H)| = 2$. However, this is a contradiction, and so X is abipodic.

(c) If X is abipodic and contains a tripod H, then $|Bd(H)| = 3$ and $P(X) = S(X)$ [12, Corollary 4]. Furthermore, Bd(H) separates X [12, Lemma 3]. Therefore $S(X) \leq |Bd(H)| = 3$. Since $S(X) \geq 3$ by part (b), then $P(X) = S(X) = 3$.

Conversely, suppose that $P(X) = S(X) = 3$. Since $P(X) = 3$, then X contains a tripod. Furthermore, X contains no k-pod for $1 < k < 3$. Thus X contains no bipod, and so X is abipodic.

Lemma 2(c) showed that in an abipodic homogeneous continuum, the existence of a tripod is equivalent to a separation number of 3. Corollary 3 establishes a related result. More specifically, any three points that separate an abipodic homogeneous continuum constitute the boundaries of complementary tripods in X [12, Theorem 6].

Corollary 3: Suppose X is an abipodic homogeneous continuum, $x,y,z \in X$, and $A|B$ is a separation of $X-\{x,y,z\}$. Then $A\cup\{x,y,z\}$ and $B\cup\{x,y,z\}$ are tripods in X with common boundary $\{x,y,z\}$.

Proof: Since X is abipodic, then $S(X) \geq 3$ by Lemma 2(b). Furthermore, since $\{x,y,z\}$ separates X then $S(X) = 3$. Consequently, since $A|B$ is a separation of $X-\{x,y,z\}$, then $A\cup\{x,y,z\}$ and $B\cup\{x,y,z\}$ are tripods in X with common boundary $\{x,y,z\}$ [12, Lemma 2].

In homogeneous continua which have pod number n, n-pods exist in pairs [12, Theorem 6]. The same is true for the existence of tripods in abipodic homogeneous continua. All that must be verified is the pod number of the space, which was established by Lemma 2.

Corollary 4: If H is a tripod in an abipodic homogeneous continuum X, then Cl(X−H) is also a tripod in X and Bd[Cl(X−H)] = Bd(H).

Proof: Since X is an abipodic homogeneous continuum and contains the tripod H, then P(X) = 3 by Lemma 2(c). Thus since H is a tripod in X, then Cl(X−H) is also a tripod in X and Bd[Cl(X−H)] = Bd(H) [12, Theorem 6].

Alternate Proof: Since H is a tripod in X then Bd(H) separates X and Int(H)|(X−H) is a separation of X−Bd(H) [12, Lemma 3]. Therefore Int(H)∪Bd(H) and (X−H)∪Bd(H) are tripods in X with common boundary Bd(H) by Corollary 3. However, (X−H)∪Bd(H) = (X−H)∪Bd(X−H) = Cl(X−H) ([8, p. 87, no. 12],[11, p. 28, Theorem 3.14(a)]). Hence Cl(X−H) is a tripod in X with Bd[Cl(X−H)] = Bd(H).

In 2007 Winton showed that if X is a homogeneous continuum with pod number P(X) = n and H is an n-pod in X, then Cl(X−H) is also an n-pod in X whose boundary is Bd(H) [12, Theorem 6]. In this case Cl(X−H) is called the complementary n-pod of H in X [12]. Henceforth, if H is a tripod in an abipodic space X, then H and Cl(X−H) will be called complementary tripods in X based on Corollary 4.

It is known that homogeneous continua cannot be separated by a single point [13, Theorem 2]. Furthermore, n-pods in homogeneous continua with pod number n ≥ 3 have the same property [13, Theorem 6]. As a special case, we have the following result.

Corollary 5: If H is a tripod in an abipodic homogeneous continuum X, then no single point in X separates H.

Proof: Since X is an abipodic homogeneous continuum and contains the tripod H, then P(X) = 3 by Lemma 2(c). Consequently no single point in X separates H [13, Theorem 6].

The following result establishes a set containment relationship between tripods in a homogeneous metric continuum with pod number 3 when an intersection property between their boundaries and interiors exists. More specifically, when the boundary of one tripod intersects the interior of another, and their respective complementary tripods have a symmetrical intersection property, then one of the original tripods is contained in the interior of the other. Furthermore, the complementary tripods have a similar relationship of one being contained in the interior of the other.

Lemma 6: Suppose that X is a homogeneous metric continuum, $P(X) = 3$, H and K are tripods in X, $H' = Cl(X-H)$, $K' = Cl(X-K)$, $Bd(K) \cap Int(H) \neq \emptyset$, and $Bd(H') \cap Int(K') \neq \emptyset$. Then $K \subseteq Int(H)$ and $H' \subseteq Int(K')$.

Proof: Since X is a homogeneous continuum, $P(X) = 3$, and H and K are tripods in X, then $H' = Cl(X-H)$ and $K' = Cl(X-K)$ are the complementary tripods to H and K, respectively, with $Bd(H') = Bd(H)$ and $Bd(K') = Bd(K)$ [12, Theorem 6]. Furthermore, since $Int(K') = K'-Bd(K')$ ([7, p. 46, Theorem 10],[11, p. 28, Theorem 3.14(b)]) then $Bd(K') \cap Int(K') = \emptyset$. However, $Bd(H') \cap Int(K') \neq \emptyset$ by the hypothesis. Consequently $Bd(H') \neq Bd(K')$, and so $Bd(H') \cap Bd(K') = \emptyset$ ([14, Lemma 4],[14, Theorem 5]). Therefore $Bd(H') \subseteq X-Bd(K') = Int(K') \cup (X-K')$ since $\{Int(K'), Bd(K'), X-K'\}$ is a partition of X ([2, p. 142, Theorem 30.2],[4, p. 72, Theorem 4.11(4)]).

Suppose $Bd(H') \cap (X-K') \neq \emptyset$. Since $Bd(H') \cap Int(K') \neq \emptyset$ by hypothesis, and it was shown above that $Bd(H') \subseteq Int(K') \cup (X-K')$, then either $Int(K')$ contains one point of $Bd(H')$ and $X-K'$ contains the other two points of $Bd(H')$, or $X-K'$ contains one point of $Bd(H')$ and $Int(K')$ contains the other two points of $Bd(H')$. In either case, $Bd(H')$ does not separate X ([14, Lemma 1],[14, Lemma 2]). This is a contradiction [12, Lemma 3], and so $Bd(H') \cap (X-K') = \emptyset$. Hence $Bd(H) = Bd(H') \subseteq Int(K') = X-K$ [8, p. 85, Theorem 4.12], so that $K \subseteq X-Bd(H) = Int(H) \cup (X-H)$ ([2, p. 142, Theorem 30.2],[4, p. 72, Theorem 4.11(4)]). Since $Int(H) \mid (X-H)$ is a separation of $X-Bd(H)$ [12, Lemma 3], then $Int(H)$ and $X-H$ are mutually separated. Consequently, since K is connected and $K \subseteq Int(H) \cup (X-H)$, then either $K \subseteq Int(H)$ or $K \subseteq X-H$ [11, p. 192, Corollary 26.6]. However, since $K \cap Int(H) \supseteq Bd(K) \cap Int(H) \neq \emptyset$ by the hypothesis, then $K \subseteq Int(H)$.

Finally, $H' \subseteq Int(K')$ by a similar argument.

It is well known that the intersection K of a nonempty nested collection S of subcontinua in a topological space X is a subcontinuum of X. Furthermore, if K has more than one point, then K contains infinitely many points. However, if X is compact and Hausdorff, and the subcontinua in S are tripods, then K contains no more than three points from the boundaries of the tripods in S. Additionally, if K has more than one point, then not only do the tripods in S have infinitely many points in common, but so do their interiors.

Theorem 7: Suppose that S is a nonempty nested collection of tripods in a compact Hausdorff space X.

(a) Then $\bigcap_{H \in S} H$ contains at most three points which are in the boundary of some tripod in S.

(b) Furthermore, if $\bigcap_{H \in S} H$ is not a singleton set then $\bigcap_{H \in S} Int(H)$ is infinite.

Proof:

(a) Assume that $\bigcap_{H\in S} H$ contains four distinct points $\{p_i\}_{i=1}^{4}$, each of which is in the boundary of some tripod in S. Then there exists $\{H_i\}_{i=1}^{4} \subseteq S$ such that $p_i \in Bd(H_i)$ for $1 \le i \le 4$. Since S is a nested collection then $\{H_i\}_{i=1}^{4}$ is linearly ordered by set inclusion. Without loss of generality, suppose that $H_i \subseteq H_{i+1}$ for $1 \le i \le 3$.

Since $\{p_i\}_{i=1}^{4} \subseteq \bigcap_{H\in S} H$ and $H_1 \in S$ then $\{p_i\}_{i=1}^{4} \subseteq H_1$. Suppose $1 \le i \le 4$, so that $H_1 \subseteq H_i$. If $p_i \notin Bd(H_1)$ then $p_i \in H_1 - Bd(H_1) = Int(H_1)$ ([7, p. 46, Theorem 10],[11, p. 28, Theorem 3.14(b)]). Furthermore, since $H_1 \subseteq H_i$ then $Int(H_1) \subseteq Int(H_i)$. Therefore $p_i \in Int(H_1) \subseteq Int(H_i) = H_i - Bd(H_i)$ ([7, p. 46, Theorem 10],[11, p. 28, Theorem 3.14(b)]), and so $p_i \notin Bd(H_i)$ for $1 \le i \le 4$. This contradicts the definition of $\{H_i\}_{i=1}^{4}$, and so $p_i \in Bd(H_1)$ for $1 \le i \le 4$. However, this is a contradiction since $|Bd(H_1)| = 3$ and $p_i \ne p_j$ whenever $1 \le i < j \le 4$. The result follows.

(b) Now suppose $\bigcap_{H\in S} H$ is not a singleton set, so that $\left|\bigcap_{H\in S} H\right| \ne 1$. Since H is compact for each $H\in S$ and X is Hausdorff, then H is closed for each $H\in S$ ([1, p. 81, Corollary 5.13],[3, p. 165, Theorem 6.4]). Furthermore, since S is a nonempty nested collection then S has the finite intersection property. Therefore $\bigcap_{H\in S} H \ne \varnothing$ since X is compact ([7, p. 136, Theorem 1],[10, p. 112, Theorem D]). Thus $\left|\bigcap_{H\in S} H\right| \ne 0$, and so $\left|\bigcap_{H\in S} H\right| > 1$. Furthermore, since S is a nonempty nested collection then S is directed by both normal and reverse set inclusion. Consequently, $\bigcap_{H\in S} H$ is a continuum [11, p. 203, Theorem 28.2], and so $\bigcap_{H\in S} H$ is infinite [5, p. 80, Theorem 5.5].

Define T to be the set of all points in $\bigcap_{H\in S} H$ which are in the boundary of some tripod in S. If $x \in \left(\bigcap_{H\in S} H\right) - T$ then $x \in H$ but $x \notin Bd(H)$ for each $H\in S$. Therefore $x \in H - Bd(H) = Int(H)$ for each $H\in S$ ([7, p. 46, Theorem 10],[11, p. 28, Theorem 3.14(b)]). Hence $x \in \bigcap_{H\in S} Int(H)$, and so $\left(\bigcap_{H\in S} H\right) - T \subseteq \bigcap_{H\in S} Int(H)$. Conversely, if $x \in \bigcap_{H\in S} Int(H)$ then $x \in Int(H) = H - Bd(H)$ for each $H\in S$

([7, p. 46, Theorem 10],[11, p. 28, Theorem 3.14(b)]). Thus $x \in H$ but $x \notin Bd(H)$ for each $H \in S$, so that $x \in \left(\bigcap_{H\in S} H\right) - T$. Therefore $\bigcap_{H\in S} Int(H) \subseteq \left(\bigcap_{H\in S} H\right) - T$, and so $\bigcap_{H\in S} Int(H) = \left(\bigcap_{H\in S} H\right) - T$. Since it was shown above that $\bigcap_{H\in S} H$ is infinite, and $|T| \leq 3$ by part (a) above, then $\bigcap_{H\in S} Int(H) = \left(\bigcap_{H\in S} H\right) - T$ is infinite.

Suppose that X is a set and S is a collection of subsets of X. The relation ~ on S defined by A~B if and only if $A \subseteq B$ for each $A,B \in S$ is called the ordering of S by inclusion [5, p. 275, Definition 1.7]. Furthermore, ~ is a partial order on S [5, p. 275, Theorem 1.8]. A modified form of ordering by inclusion will now be introduced for an arbitrary collection S of subsets of topological space. It will be shown that this relation, which is based on the topology of the space, also partially orders S.

Lemma 8: Suppose X is a topological space, S is a collection of subsets of X, and ~ is the relation on S defined by A~B if and only if $A \subseteq Int(B)$ or $A = B$ for each $A,B \in S$. Then ~ is a partial order on S.

Proof: Clearly A~A for each $A \in S$. Therefore ~ is reflexive on S.

Suppose $A,B \in S$, A~B, and B~A. If $A \subseteq Int(B)$ and $B \subseteq Int(A)$ then $A \subseteq Int(B) \subseteq B \subseteq Int(A) \subseteq A$, so that $A = B$. Otherwise $A = B$ immediately since A~B and B~A. Thus in either case we have $A = B$, and so ~ is antisymmetric on S.

Now suppose that $A,B,C \in S$, A~B, and B~C. If $A \subseteq Int(B)$ and $B \subseteq Int(C)$ then $A \subseteq Int(B) \subseteq B \subseteq Int(C)$. Furthermore, if $A = B$ and $B = C$, then clearly $A = C$. If $A = B$ and $B \subseteq Int(C)$ then $A \subseteq Int(C)$. Similarly, if $A \subseteq Int(B)$ and $B = C$ then $A \subseteq Int(C)$. Thus in all possible cases either $A \subseteq Int(C)$ or $A = C$, so that A~C, and so ~ is transitive on S. Hence ~ is a partial order on S.

The partial order established in Lemma 8 will be critical for the main theorem. Hence we have the following definition.

Definition 9: Suppose X is a topological space and S is a collection of subsets of X. The relation ~ on S defined by A~B if and only if $A \subseteq Int(B)$ or $A = B$ for each $A,B \in S$ is called the ordering of S by interior inclusion.

The ordering by interior inclusion established in Lemma 8 and formalized in Definition 9 will now be employed to produce the main result of the paper. Specifically, the partial order by interior inclusion will be applied, along with Kuratowski's Lemma, to guarantee the existence of a maximal chain of tripods in X. This maximal chain and the homogeneity of the space with then be used to show that in a homogeneous metric continuum X with pod number 3, each point is the result of intersecting a chain of tripods which is maximal in the set of all tripods in X relative to ordering by interior inclusion.

Main Theorem

Theorem 10: Suppose X is a homogeneous metric continuum, P(X) = 3, and S is the collection of all tripods in X. Then each point in X is the intersection of a maximal chain of tripods in S relative to the ordering of S by interior inclusion.

Proof: By Lemma 8 and Definition 9 the ordering ~ of S by interior inclusion is a partial order on S. Since P(X) = 3 then X contains a tripod W, so that {W} is a chain in S. Thus by Kuratowski's Lemma [7, p. 33, Theorem 25(d)], there is a maximal chain M of tripods in S relative to ~ which contains {W}. Therefore M is nonempty since $W \in M$. Furthermore, since M is a chain relative to ~, then M is nested.

Assume that $\bigcap_{H \in M} H$ is not a singleton set. Therefore $\bigcap_{H \in M} \mathrm{Int}(H)$ is infinite by Theorem 7(b), so there exists some $x \in \bigcap_{H \in M} \mathrm{Int}(H)$. Since W is a tripod then there exist $p,q,r \in X$ such that Bd(W) = {p,q,r}. Furthermore, since X is homogeneous then there is a homeomorphism $h: X \to X$ for which h(p) = x. Define y = h(q) and z = h(r). Since {p,q,r} separates X [12, Lemma 3] and h is a homeomorphism, then {x,y,z} also separates X. Hence there is a separation $A|B$ of $X - \{x,y,z\}$. Since X contains the tripod W and P(X) = 3, then S(X) = 3 as well [12, Corollary 4]. Therefore $K = A \cup \{x,y,z\}$ and $K' = B \cup \{x,y,z\}$ are complementary tripods in X with Bd(K) = Bd(K′) = {x,y,z} [12, Lemma 2]. Furthermore, A = Int(K) and B = X−K = Int(K′) [12, Lemma 3].

Suppose that $H \in M$, so that $x \in \mathrm{Int}(H)$. Since $x \in \mathrm{Bd}(K) = \mathrm{Bd}(K')$ as well, then $\mathrm{Bd}(K) \cap \mathrm{Int}(H) = \mathrm{Bd}(K') \cap \mathrm{Int}(H) \neq \varnothing$. Furthermore, since P(X) = 3 then X is abipodic, and so $H' = \mathrm{Cl}(X - H)$ is also a tripod in X with Bd(H′) = Bd(H) (Corollary 4,[12, Theorem 6]). Since $x \in \mathrm{Int}(H) = H - \mathrm{Bd}(H)$ ([7, p. 46, Theorem 10],[11, p. 28, Theorem 3.14(b)]) and $x \in \mathrm{Bd}(K)$ then $\mathrm{Bd}(K) \neq \mathrm{Bd}(H) = \mathrm{Bd}(H')$. Therefore $\mathrm{Bd}(H') \cap \mathrm{Bd}(K) = \varnothing$ ([14, Lemma 4],[14, Theorem 5]), so that $\mathrm{Bd}(H') \subseteq X - \mathrm{Bd}(K) = X - \{x,y,z\} = A \cup B = \mathrm{Int}(K) \cup \mathrm{Int}(K')$ ([2, p. 142, Theorem 30.2],[4, p. 72, Theorem 4.11(4)]). Consequently either $\mathrm{Bd}(H') \cap \mathrm{Int}(K) \neq \varnothing$ or $\mathrm{Bd}(H') \cap \mathrm{Int}(K') \neq \varnothing$.

Suppose that $\mathrm{Bd}(H') \cap \mathrm{Int}(K) \neq \varnothing$. Since it was shown above that $\mathrm{Bd}(K') \cap \mathrm{Int}(H) \neq \varnothing$ as well, then $K' \subseteq \mathrm{Int}(H)$ and $H' \subseteq \mathrm{Int}(K)$ by Lemma 6. On the other hand, suppose that $\mathrm{Bd}(H') \cap \mathrm{Int}(K') \neq \varnothing$. Since it was shown above that $\mathrm{Bd}(K) \cap \mathrm{Int}(H) \neq \varnothing$ as well, then $K \subseteq \mathrm{Int}(H)$ and $H' \subseteq \mathrm{Int}(K')$ by Lemma 6. Thus either $K \subseteq \mathrm{Int}(H)$ or $K' \subseteq \mathrm{Int}(H)$ for each $H \in M$.

Assume that neither K nor K′ is contained in $\bigcap_{H \in M} \mathrm{Int}(H)$. Then there exist $H_1, H_2 \in M$ such that $K \not\subseteq \mathrm{Int}(H_1)$ and $K' \not\subseteq \mathrm{Int}(H_2)$. However, since either

$K \subseteq Int(H)$ or $K' \subseteq Int(H)$ for each $H \in M$, then $K' \subseteq Int(H_1)$ and $K \subseteq Int(H_2)$. Moreover, since M is a chain and $H_1, H_2 \in M$ then either $H_1 \sim H_2$ or $H_2 \sim H_1$. If $H_1 \sim H_2$ then $H_1 \subseteq Int(H_2)$ or $H_1 = H_2$, so that $Int(H_1) \subseteq Int(H_2)$. Therefore $K' \subseteq Int(H_1) \subseteq Int(H_2)$, a contradiction. On the other hand, if $H_2 \sim H_1$ then $H_2 \subseteq Int(H_1)$ or $H_2 = H_1$, so that $Int(H_2) \subseteq Int(H_1)$. Thus $K \subseteq Int(H_2) \subseteq Int(H_1)$, which is again a contradiction. Thus either K or K′ is contained in $\bigcap_{H \in M} Int(H)$.

Without loss of generality, suppose $K \subseteq \bigcap_{H \in M} Int(H)$. Then $K \subseteq Int(H)$ for each $H \in M$, so that K~H for each $H \in M$. However, since $K \subseteq Int(H) = H - Bd(H)$ for each $H \in M$ ([7, p. 46, Theorem 10],[11, p. 28, Theorem 3.14(b)]), then $K \cap Bd(H) = \emptyset$, and so $K \neq H$ for each $H \in M$. Consequently, $M \cup \{K\}$ is a chain in S relative to ~ which properly contains M. However, this contradicts the fact that M is a maximal chain in S relative to ~.

Therefore $\bigcap_{H \in M} H$ is a singleton set, so that $\bigcap_{H \in M} H = \{t\}$ for some $t \in X$. Thus {t} is the intersection of a maximal chain of tripods in S relative to the ordering of S by interior inclusion. Since X is homogeneous, then each point in X is such an intersection.

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