PTOLEMY'S THEOREM – A New Proof

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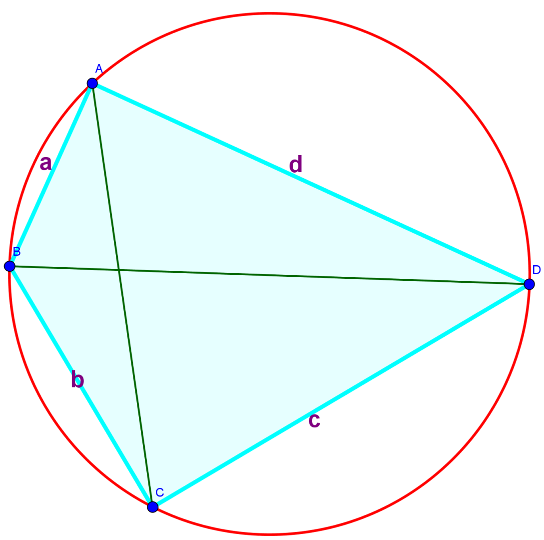
Abstract*:* In this article we present a new proof of Ptolemy's theorem using a metric relation of circumcenter in a different approach..

Keywords: Ptolemy's theorem, Circumcenter, Cyclic Quadrilateral.

1. INTRODUCTION

The classical theorem of Ptolemy states that if A, B, C, D are, in this order, four points on the circle O, then $AC.BD = AB.CD + AD.BC$.

In the literature of Euclidean geometry there are many proofs for this celebrated theorem (some of them can be found in [3], [4],[7],[8], [9] and [10]). In the article [2] we presented a proof of this theorem using a lemma related to the metric relation of circumcenter, but now in our present paper we will use the same metric relation but in a different approach. Our proof actually follows as first we will prove a lemma using the metric relation on circumcenter, using this lemma we will prove a new generalization of ptolemy's theorem, based on this new generalization we will prove ptolemy's first and second theorems which are special cases of the generalization.



2. Some Basic Lemma's

Lemma-1

Let A, B, C and D are the angles of a cyclic quadrilateral such that
$\angle DAC = A_1, \angle CAB = A_2, \angle ABD = B_1, \angle DBC = B_2,$
$\angle BCA = C_1, \angle ACD = C_2, \angle CDB = D_1, \angle BDA = D_2$ *and*
if R is the circumradius then

(1.1).

$$\sin 2A_1 + \sin 2A_2 - \sin 2A = 4\sin A_1 \sin A_2 \sin A = \frac{BC.CD.BD}{2R^3}$$

(1.2).

$$\sin 2B_1 + \sin 2B_2 - \sin 2B = 4\sin B_1 \sin B_2 \sin B = \frac{AD.CD.AC}{2R^3}$$

(1.3).

$$\sin 2C_1 + \sin 2C_2 - \sin 2C = 4\sin C_1 \sin C_2 \sin C = \frac{AB.AD.BD}{2R^3}$$

(1.4).

$$\sin 2D_1 + \sin 2D_2 - \sin 2D = 4\sin D_1 \sin D_2 \sin D = \frac{AB.BC.AC}{2R^3}$$

Proof:

Using the fact $\sin P + \sin Q = 2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}$ and
$\sin P - \sin Q = 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}$

Using transformation of the angles It is easy to verify that

$$\sin 2A_1 + \sin 2A_2 - \sin 2A = 4\sin A_1 \sin A_2 \sin A$$
………(Ω)

Now using sine rule for triangles ΔABC and ΔADC, We have $\sin A_2 = \frac{BC}{2R}$, $\sin A_1 = \frac{CD}{2R}$ and

$$\sin\left(A_1 + A_2\right) = \sin A = \frac{BD}{2R}$$

By replacing these values in (Ω) and by little algebra we get desired (1.1),
In the similar manner we can prove (1.2), (1.3) and (1.4)

Lemma-2

If S is the circumcenter of an acute or right triangle and M be any point in the plane of triangle then

$$SM^2 = \frac{R^2}{2\Delta}\left(\sin 2A.AM^2 + \sin 2B.BM^2 + \sin 2C.CM^2 - 2\Delta\right)$$

Proof:

The proof of above lemma can be found in [1]

3. MAIN RESULTS

NEW GENERALIZATION OF PTOLEMYS THEOREM

Let ABCD is a cyclic quadrilateral whose circumcenter is S and R is its circumraduis, If M be any point in the plane of the quadrilateral then

(2.1).

$$\left(\sin 2A_1 + \sin 2A_2 - \sin 2A\right)AM^2 + \left(\sin 2C_1 + \sin 2C_2 - \sin 2C\right)CM^2$$

$$= \left(\sin 2B_1 + \sin 2B_2 - \sin 2B\right)BM^2 + \left(\sin 2D_1 + \sin 2D_2 - \sin 2D\right)DM^2$$

(2.2). $$\frac{AC}{BD} = \frac{BC.CD.AM^2 + AB.AD.CM^2}{AD.CD.BM^2 + AB.BC.DM^2}$$

Proof:

Using lemma-2 we have

$$SM^2 = \frac{R^2}{2\Delta}\left(\sin 2A.\mathrm{AM}^2 + \sin 2B.\mathrm{BM}^2 + \sin 2C.\mathrm{CM}^2 - 2\Delta\right)$$

It can be rewritten as

$$\frac{2\Delta}{R^2}\left(SM^2+R^2\right)=\sin 2A.\mathrm{AM}^2+\sin 2B.\mathrm{BM}^2+\sin 2C.\mathrm{CM}^2$$

…….......(π)

Now since S and R be the circumcenter and circumraduis of quadrilateral ABCD, so the circumcenter and circumradius of triangles ΔABC, ΔBCD, ΔCDA and ΔDAB are S , R

And let area of $\Delta ABC=\Delta_1$, area of $\Delta BCD=\Delta_2$, area of $\Delta CDA=\Delta_3$ and area of $\Delta DAB=\Delta_4$.

It is clear that area of quadrilateral ABCD $=\Delta=\Delta_1+\Delta_3=\Delta_2+\Delta_4$

Now by applying (1.1) for the triangles ΔABC, ΔBCD, ΔCDA and ΔDAB successively we get,

$$\frac{2\Delta_1}{R^2}\left(SM^2+R^2\right)=\sin 2A_2.\mathrm{AM}^2+\sin 2B.\mathrm{BM}^2+\sin 2C_1.\mathrm{CM}^2$$

………(€₁)

$$\frac{2\Delta_2}{R^2}\left(SM^2+R^2\right)=\sin 2B_2.\mathrm{BM}^2+\sin 2D_1.\mathrm{DM}^2+\sin 2C.\mathrm{CM}^2$$

……….(€₂)

$$\frac{2\Delta_3}{R^2}\left(SM^2+R^2\right)=\sin 2C_2.\mathrm{CM}^2+\sin 2D.\mathrm{DM}^2+\sin 2A_1.\mathrm{AM}^2$$

……...(€₃)

$$\frac{2\Delta_4}{R^2}\left(SM^2+R^2\right)=\sin 2B_1.\mathrm{BM}^2+\sin 2D_2.\mathrm{DM}^2+\sin 2A.\mathrm{AM}^2$$

……..(€₄)

Now by (€$_1$)+(€$_3$)-(€$_2$)-(€$_4$) and also using the fact

$=\Delta=\Delta_1+\Delta_3=\Delta_2+\Delta_4$

We get conclusion (2.1),

Now using (2.1) and lemma-1(1.1), (1.2), (1.3), (1.4)

We can prove conclusion (6).

PTOLEMY'S THEOREM

Let ABCD be any cyclic quadrilateral such that AC and BD are its diagonals then

(3.1). $AC.BD = AB.CD + AD.BC$ *(Ptolemy's First Theorem)*

(3.2). $\dfrac{AC}{BD} = \dfrac{BC.CD + AB.AD}{AD.CD + AB.BC}$ *(Ptolemy's Second Theorem)*

Proof:

Now from conclusion (2.2)

We have $\dfrac{AC}{BD} = \dfrac{BC.CD.AM^2 + AB.AD.CM^2}{AD.CD.BM^2 + AB.BC.DM^2}$

Since (2.2) is true for any M, So as to prove (3.1) fix M as either A or B or C or D and to prove (3.2) fix M as S(circumcenter) where as we can use SA=SB=SC=SD=R for simplification. Hence Ptolemy's Theorem is proved. For historical studies and further generalization of Ptolemy's Theorem refer [5], [6], [11], [12], [13] and [14].

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